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Orthogonal testing families and holomorphic extension from the sphere to the ball

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Abstract

Let $\mathbb{B}^2$ denote the open unit ball in $\mathbb{C}^2$, and let $p \in \mathbb{C}^2 \setminus \overline{\mathbb{B}^2}$. We prove that if f is an analytic function on the sphere $\partial\mathbb{B}^2$ that extends holomorphically in each variable separately and along each complex line through p, then f is the trace of a holomorphic function in the ball.

Keywords Analytic discs · Holomorphic extension · Testing families

Mathematics Subject Classification Primary 32V25 · Secondary 32V20 · 32V40

1 Introduction and main theorem

It is a well-known fact in the theory of several complex variables that a function is holomorphic if and only if it is holomorphic in each variable separately. This result goes back to Hartogs [13]. It is natural to consider a boundary version of Hartogs' theorem. The general problem is to take a boundary function and ask if holomorphic extensions on vertical and horizontal slices are enough to guarantee an extension which is holomorphic in both variables simultaneously. In [14] Lawrence proved that vertical and horizontal slices are enough to detect the existence of holomorphic extension to the interior for functions defined on a small perturbation of the boundary of the unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$. However, the result is not true for the ball itself, for which additional conditions are needed.

There is a vast literature on describing families of directions which suffice for testing analytic extension of a continuous function f from the sphere to the ball. The first significant result was obtained by Stout [18], who used as testing family all the straight lines. Reducing

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the testing family, Agranovsky and Semenov [2] used the lines which meet an open subset of the ball, Rudin [17] the lines tangent to a concentric subsphere, Baracco, Tumanov and Zampieri [4] the lines tangent to any strictly convex subset of $\mathbb{B}^2$. Among the many contributions to the problem we mention [1,5,9,10,15,20].

It is well known that the lines which meet a single point do not suffice. With additional hypotheses on the initial regularity of f on $\partial\mathbb{B}^2$ (namely, for f analytic rather than just continuous) one can prove that the families of lines through the following sets of points do suffice: two interior points (Agranovsky [3]), one boundary point (Baracco [7]), two points in $\mathbb{C}^2 \setminus \overline{\mathbb{B}^2}$ whose joining line is tangent to the sphere (Baracco and Pinton [8]).

In [6] Baracco proved, for f continuous on $\partial\mathbb{B}^2$, that three non-aligned points in the ball suffice. The result was later improved by Globevnik [11], who allowed the points to lie outside $\mathbb{B}^2$ provided that at least one of the joining lines meets the ball. At the end of his paper, Globevnik asked the following question: let $a, b, c \in \mathbb{C}^2$ be three points whose joining lines do not meet the ball. Do the lines through a, b and c constitute a testing family for holomorphic extension?

In this paper, we give a partial answer to Globevnik's question, under the assumption that f is analytic on $\partial\mathbb{B}^2$. Here is our main result.

Theorem 1.1 *Let f be an analytic function on the sphere $\partial\mathbb{B}^2$ which extends holomorphically in each variable separately and along each complex line through a point $p \in \mathbb{C}^2 \setminus \overline{\mathbb{B}^2}$. Then f extends holomorphically to $\mathbb{B}^2$.*

We will only prove the case $p = (p_1, p_2)$, $|p_1| > 1$, $|p_2| > 1$, which falls into Globevnik's question, with two of the points being at infinity. We will not deal with the cases $|p_1| < 1$ or $|p_2| < 1$, which were already treated by Globevnik [11].

For the proof of our theorem we employ techniques related to stationary discs in the sense of Lempert [16] that have already been used in this context in [4,6,7]. We add a better understanding of the geometry of the space of lifts of stationary discs and the use of a continuity principle.

2 Stationary discs

In this section we summarize some basic facts on stationary discs and we prove a technical lemma that will be used in the proof of Theorem 1.1. For more background information on analytic and stationary discs, we refer the reader to the original paper of Lempert [16] and Tumanov's lecture notes [19].

Let M be a smooth real manifold in $\mathbb{C}^n$ and let TM denote its tangent bundle. For $p \in M$ recall the space $T_p^{1,0}M \subset T_pM \otimes \mathbb{C}$ of complex $(1,0)$-vectors defined as

$$T_p^{1,0}M := \left\{X \in T_pM \otimes \mathbb{C}\colon X = \sum a_j\, \partial/\partial z_j\right\}.$$

Let $T^*\mathbb{C}^n$ be the real cotangent bundle of $\mathbb{C}^n$. Since every $(1,0)$-form is uniquely determined by its real part, we represent $T^*\mathbb{C}^n$ as the space of $(1,0)$-forms on $\mathbb{C}^n$. More precisely, for $z \in \mathbb{C}^n$, we use the identification

$$T_z^*\mathbb{C}^n \simeq \left(T_z^{1,0}\mathbb{C}^n\right)^*,$$
$$\omega \xrightarrow{\sim} \Omega.$$

where $\langle\omega, X\rangle = \operatorname{Re}\langle\Omega, X\rangle$ for all $X \in T_z\mathbb{C}^n$. Let $T_M^*\mathbb{C}^n \subset T^*\mathbb{C}^n$ be the real *conormal bundle* of M. Using the representation of $T^*\mathbb{C}^n$ by $(1,0)$-forms, we define the fiber $(T_M^*\mathbb{C}^n)_p$ at $p \in M$ as

$$\left(T_M^*\mathbb{C}^n\right)_p := \left\{\omega \in T_p^*\mathbb{C}^n : \operatorname{Re}\omega|_{T_pM} = 0\right\}.$$

Note that if r is a defining function for M then the conormal bundle $T_M^*\mathbb{C}^n$ is generated by $\partial r = \sum \partial r/\partial z_j dz_j$.

Let Δ be the unit disc in $\mathbb{C}$. An *analytic disc* in a complex manifold X is a holomorphic map $A : \Delta \to X$. We say that A is *attached* to some set $M \subset X$ if A is continuous in the closed disc $\overline{\Delta}$ and $A(\partial\Delta) \subset M$.

Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$. An analytic disc A attached to ∂D is said to be *stationary* if there exists a map $\lambda\colon \partial\Delta \to \mathbb{R}_{>0}$ such that the function $\tau\lambda(\tau)\partial r(A(\tau))$, defined for $\tau \in \partial\Delta$, extends to a function continuous in $\overline{\Delta}$ and holomorphic in Δ. In other words, a disc is stationary if it admits a meromorphic "lift" to a disc in the cotangent bundle attached to the conormal bundle.

Let $\mathbb{B}^n$ be the open unit ball in $\mathbb{C}^n$. It is immediate to verify that the conormal bundle of the n-sphere $\partial\mathbb{B}^n$ is given by

$$T^*_{\partial\mathbb{B}^n}\mathbb{C}^n = \left\{(z, \lambda\overline{z}),\, z \in \partial\mathbb{B}^n,\ \lambda \in \mathbb{R}\right\}.$$

In this case, the stationary discs are precisely the straight ones, that is, the ones obtained by intersecting the ball with complex lines.

The following two propositions are well known.

Proposition 2.1 *Let $A\colon \Delta \to X$ be a stationary disc and let $\varphi\colon \Delta \to \Delta$ be an automorphism of the unit disc. Then $A \circ \varphi\colon \Delta \to X$ is also stationary.*

Proof Let φ be given by

$$\varphi(\tau) = \alpha\,\frac{\tau - a}{1 - \tau\overline{a}},$$

for some $a, \alpha \in \mathbb{C}$ with $|\alpha| = 1$, $|a| < 1$. The proposition is proved if we can find a map $\widetilde{\lambda}\colon \partial\Delta \to \mathbb{R}_{>0}$ such that the function $\tau\widetilde{\lambda}(\tau)\partial r(A(\varphi(\tau)))$, defined for $\tau \in \partial\Delta$, extends to a function continuous in $\overline{\Delta}$ and holomorphic in Δ. For $\tau \in \partial\Delta$, let

$$\widetilde{\lambda}(\tau) := |\tau - a|^2\lambda(\varphi(\tau)).$$

On $\partial\Delta$ we have

$$\tau\widetilde{\lambda}(\tau)\partial r(A(\varphi(\tau))) = (1 - \tau\overline{a})^2\,\underbrace{\frac{(\tau - a)}{(1 - \tau\overline{a})}\lambda(\varphi(\tau))A(\varphi(\tau))}_{H(\tau)}.$$

Since A is stationary, $H(\tau)$ extends to a function in $\overline{\Delta}$ holomorphic in Δ, and the same is true for $(1 - \tau\overline{a})^2 H(\tau)$. □

Proposition 2.2 *Let $\lambda_1, \lambda_2\colon \partial\Delta \to \mathbb{R}_{>0}$ be such that both $\tau\lambda_1(\tau)\partial r(A(\tau))$ and $\tau\lambda_2(\tau)\partial r(A(\tau))$ extend to functions continuous in $\overline{\Delta}$ and holomorphic in Δ. Assume also $\lambda_1(1) = \lambda_2(1)$. Then $\lambda_1 = \lambda_2$.*

Proof Since ∂r generates the conormal bundle, we have

$$\operatorname{Re}\left\langle \partial r(A), \partial_\theta A\left(e^{i\theta}\right)\right\rangle = 0.$$

Therefore

$$\operatorname{Re}\left\langle (\lambda_1 - \lambda_2)\,\partial r(A), ie^{i\theta}A'\left(e^{i\theta}\right)\right\rangle = 0. \tag{2.1}$$

Equation (2.1) implies that the holomorphic function $\langle(\lambda_1-\lambda_2)\partial r(A), ie^{i\theta}A'(e^{i\theta})\rangle$ is constant, and therefore identically zero (since it vanishes at 1). Hence

$$(\lambda_1-\lambda_2)\left\langle\partial r(A), i\tau A'(\tau)\right\rangle \equiv 0.$$

By strong pseudoconvexity of D and the Hopf Lemma, $\langle\partial r(A), i\tau A'(\tau)\rangle$ is nonvanishing on $\partial\Delta$. Hence $\lambda_1=\lambda_2$. □

The discussion above shows that the lift of a stationary disc is unique up to multiplication by a scalar function. It is therefore natural to think of a lift as a geometric object in the projective space $\mathbb{P}T^*\mathbb{C}^n$. In the rest of the paper we will use $[\,,\,]$ to denote projective coordinates.

We now restrict our attention to $\mathbb{C}^2$. In particular, for the proof of Theorem 1.1 we will need an explicit formula for the union of the lifts of the discs obtained by intersecting the unit ball $\mathbb{B}^2$ with the complex lines through a point. Let us fix some notation. For a given point $p\in\mathbb{C}^2\setminus\overline{\mathbb{B}^2}$, we consider the family of complex lines through p. For each $z\in\mathbb{B}^2$ let A_z be the disc obtained by intersecting $\mathbb{B}^2$ with the line through p and z. Each such disc A_z is stationary. We denote by A_z^* the corresponding lift in the (projectivized) cotangent bundle. The next lemma gives a precise description of the set of all lifts $M_p:=\bigcup_{z\in\mathbb{B}^2}A_z^*$.

Lemma 2.3 *For $p\in\mathbb{C}^2\setminus\overline{\mathbb{B}^2}$ the following holds*:

$$M_p=\left\{\left(z;\left[\overline{z}(z\cdot\overline{p}-1)+\overline{p}\left(1-|z|^2\right)\right]\right)\in\mathbb{B}^2\times\mathbb{P}T^*\mathbb{C}^2\right\}.$$

Proof For $z\in\mathbb{B}^2$, the complex line through z and p consists of points of the form $z+\zeta(p-z)$, $\zeta\in\mathbb{C}$. By intersecting with the unit ball, we find that the points of the disc A_z are the ones for which ζ satisfies

$$|\zeta|^2+2\mathrm{Re}\left(\zeta\frac{(p-z)\cdot\overline{z}}{|p-z|^2}\right)+\frac{|z|^2-1}{|p-z|^2}\leq 0.$$

Let $R_z:=\sqrt{\frac{|p|^2+|z|^2+|p\cdot\overline{z}|^2-|z|^2|p|^2-2\mathrm{Re}(p\cdot\overline{z})}{|z-p|^4}}$ and $C_z:=-\frac{z\cdot\overline{(p-z)}}{|p-z|^2}$. We can then parametrize A_z over the unit disc by

$$A_z(\tau)=z+(R_z\tau+C_z)(p-z),\quad \tau\in\Delta. \tag{2.2}$$

Note that R_z and C_z satisfy the following relations:

$$R_z^2=-\frac{|z|^2-1}{|p-z|^2}+\frac{\left|p\cdot\overline{z}-|z|^2\right|^2}{|p-z|^4}, \tag{2.3}$$

and

$$-R_z^2+|C_z|^2=\frac{|z|^2-1}{|p-z|^2}\,. \tag{2.4}$$

Moreover $A_z\left(-\frac{C_z}{R_z}\right)=z$. From (2.2) we can see that the meromorphic lift attached to the conormal bundle is given by

$$A_z^*(\tau)=\left(A_z(\tau),\frac{\overline{z}\tau+(R_z+\overline{C_z}\tau)(\overline{p-z})}{\tau}\right).$$

Using (2.3) and (2.4) we obtain the following formula in the projectivized cotangent bundle:

$$A_z^*\Big(-\frac{C_z}{R_z}\Big)=\left(z;\left[\overline{z}(z\cdot\overline{p}-1)+\overline{p}\left(1-|z|^2\right)\right]\right).$$

This concludes the proof. □

Remark 2.4 M_p is a manifold of dimension 4 foliated by complex curves. With a standard computation one can also see that M_p is a CR manifold of dimension 1 at all points except for those that project over the complex line $\{z \in \mathbb{C}^2 : z \cdot \overline{p} = 1\}$. We thus have a decomposition

$$M_p = M_p^{reg} \cup M_p^{sing}, \text{ where } M_p^{sing} = \{(z, [\overline{p}]) : z \cdot \overline{p} = 1\}.$$

Note that the set of CR singular points of M_p is a complex curve which intersects transversally each A_z^*.

Remark 2.5 A decomposition analogous to the one described in Remark 2.4 holds even when p is a point at infinity, that is, when we are considering all the lines parallel to a given direction. As an example, let us describe the case of the lines parallel to the z_2-direction. We denote by $(0, 1)_\infty$ the corresponding point at infinity. In this situation, we can use $z_1 \in \Delta$ as a parameter: for each z_1 we have the disc $A_{(z_1,0)}(\tau) = (z_1, \sqrt{1 - |z_1|^2}\tau)$, which lifts to

$$\begin{aligned} A^*_{(z_1,0)}((\tau) &= \left(\left(z_1, \sqrt{1-|z_1|^2}\tau\right); \left[\overline{z_1}\tau, \sqrt{1-|z_1|^2}\right]\right), \\ &= \left(\left(z_1, \sqrt{1-|z_1|^2}\tau\right); \left[\overline{z_1}\tau\sqrt{1-|z_1|^2}, 1-|z_1|^2\right]\right). \end{aligned}$$

We conclude that

$$M_{(0,1)_\infty} = \left\{\left((z_1, z_2); \left[\overline{z_1}z_2, 1-|z_1|^2\right]\right) \in \mathbb{B}^2 \times \mathbb{P}T^*\mathbb{C}^2\right\}.$$

Note that $M_{(0,1)_\infty}$ is a CR manifold of CR dimension 1 at all the points for which $z_1 \neq 0$. As before, we have a decomposition

$$M_{(0,1)_\infty} = M^{reg}_{(0,1)_\infty} \cup M^{sing}_{(0,1)_\infty}, \text{ where } M^{sing}_{(0,1)_\infty} = \{((z_1, 0); [0, 1]) : z_1 \in \Delta\}.$$

Similar formulas hold for the point at infinity $(1, 0)_\infty$, which corresponds to the lines parallel to the z_1-direction. In particular, one can check that

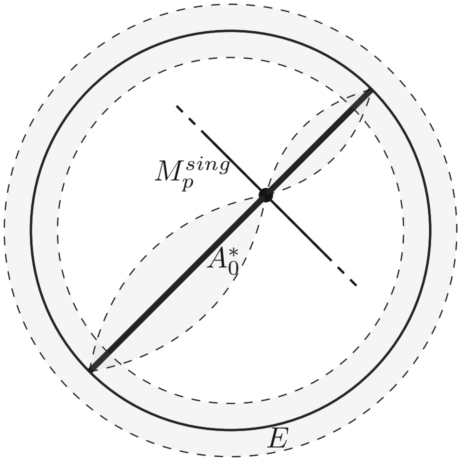
$$M_{(1,0)_\infty} = \left\{\left((z_1, z_2); \left[1-|z_2|^2, z_1\overline{z_2}\right]\right) \in \mathbb{B}^2 \times \mathbb{P}T^*\mathbb{C}^2\right\}.$$

3 Proof of Theorem 1.1

Proof Let $E := \mathbb{P}T^*_{\partial\mathbb{B}^2}\mathbb{C}^2$. The function f lifts naturally to a real analytic function $F : E \to \mathbb{C}$. Note that E is maximally totally real in $\mathbb{C}^2 \times \mathbb{P}^1_{\mathbb{C}}$, hence F extends holomorphically to a neighborhood of E. From now on, we denote by ζ the projective coordinate. By the hypotheses on the holomorphic extendibility of f, it is possible to lift f to a function defined on the manifolds $M_{(1,0)_\infty}$, $M_{(0,1)_\infty}$ and M_p. Note in fact that the lifts A_z^* in these three families do not intersect outside of E. Our goal is now to show that F extends to a holomorphic function in $\mathbb{B}^2 \times \mathbb{P}^1_{\mathbb{C}}$, and therefore F is constant in ζ. We can then conclude that F projects down to a holomorphic function in $\mathbb{B}^2$ that extends f holomorphically to the ball.

We start by observing that the function F is CR on the regular part of the three manifolds $M_{(1,0)_\infty}$, $M_{(0,1)_\infty}$ and M_p. Since M_p^{reg} is foliated by complex curves and each curve has some points where F extends holomorphically to a full neighborhood, we can apply the propagation theorem of Hanges and Treves [12] to conclude that F extends holomorphically to a neighborhood of M_p^{reg}. We will focus on the disc A_0^* in M_p, that is, the lift of the disc through p and the origin. Figure 1 illustrates the situation. There, the circle represents the projectivized conormal bundle E, and the diameter is A_0^*. The shaded regions correspond to the neighborhoods where we know F extends holomorphically.

Fig. 1 Representation of the projections onto $\mathbb{C}^2$ of E, M_p^{sing} and A_0^*



The next step is to achieve holomorphic extension at the point of CR singularity (shown in Fig. 1). To this end, we construct a continuous family $\{B_t\}$ of analytic discs in $\mathbb{P}T^*\mathbb{C}^2$ attached to $M^{reg}_{(1,0)_\infty} \cup M^{reg}_{(0,1)_\infty}$ such that:

- for some value of t, the center of B_t is at the point of CR singularity;
- for some value of t, the disc B_t is contained in the neighborhood of E where F is holomorphic.

Note that F is holomorphic on the boundary of B_t for all t, since the discs are attached to $M^{reg}_{(1,0)_\infty} \cup M^{reg}_{(0,1)_\infty}$. Hence, assuming that we have such a family of discs, we can apply the continuity principle to conclude that F extends holomorphically in a full neighborhood of A_0^*. We now consider the (continuous) family of lines through 0. Again by the continuity principle, F extends holomorphically along the lift of each disc through 0. Consequently, F is holomorphic in a neighborhood of $\{0\} \times \mathbb{P}^1_{\mathbb{C}}$, which is what we wanted to prove.

The rest of the proof is entirely devoted to constructing a family of discs $\{B_t\}$ with the properties described above. We start by recalling from Remark 2.5 the formulas:

$$M_{(1,0)_\infty} = \left\{\left((z_1, z_2); \left[1 - |z_2|^2, z_1\overline{z_2}\right]\right) \in \mathbb{B}^2 \times \mathbb{P}T^*\mathbb{C}^2\right\} \tag{3.1}$$

$$M_{(0,1)_\infty} = \left\{\left((z_1, z_2); \left[z_2\overline{z_1}, 1 - |z_1|^2\right]\right) \in \mathbb{B}^2 \times \mathbb{P}T^*\mathbb{C}^2\right\}. \tag{3.2}$$

The set E, which is the only part shared by these two manifolds, is given by

$$\left(re^{i\eta_1}, \sqrt{1-r^2}e^{i\eta_2}; \left[re^{-i\eta_1}, \sqrt{1-r^2}e^{-i\eta_2}\right]\right) \quad 0 \le r \le 1, \quad 0 \le \eta_1, \eta_2 \le 2\pi.$$

Dividing out by the last term, we now introduce a complex coordinate on the projective component. For $r \neq 1$ we then have a parametrization of E given by

$$\left(re^{i\eta_1}, \sqrt{1-r^2}e^{i\eta_2}, \frac{r}{\sqrt{1-r^2}}e^{i(\eta_2-\eta_1)}\right) \quad 0 < r < 1, \quad 0 \le \eta_1, \eta_2 \le 2\pi. \tag{3.3}$$

Equations (3.3) and (3.1) imply that

$$\left(r\rho_1e^{i\eta_1}, \sqrt{1-r^2}e^{i\eta_2}, \frac{r^2\sqrt{1-r^2}e^{i\eta_2}}{r\rho_1e^{i\eta_1}\left(1-r^2\right)}\right) \quad 0 < r, \rho_1 < 1, \quad \eta_1, \eta_2 \in \mathbb{R}, \tag{3.4}$$

is a parametrization for (almost) all $M^{reg}_{(1,0)_\infty}$. Analogously, from (3.3) and (3.2),

$$\left(re^{i\eta_1}, \sqrt{1-r^2}\rho_2 e^{i\eta_2}, \frac{r^2\sqrt{1-r^2}\rho_2 e^{i\eta_2}}{re^{i\eta_1}(1-r^2)}\right) \quad 0 < r, \rho_2 < 1, \quad \eta_1, \eta_2 \in \mathbb{R}, \tag{3.5}$$

is a parametrization for $M^{reg}_{(0,1)_\infty}$. Equations (3.4) and (3.5) together give the parametrization

$$\phi(r, \rho_1, \eta_1, \rho_2, \eta_2) = \left(r\rho_1 e^{i\eta_1}, \sqrt{1-r^2}\rho_2 e^{i\eta_2}, \frac{r^2\sqrt{1-r^2}\rho_2 e^{i\eta_2}}{r\rho_1 e^{i\eta_1}(1-r^2)}\right)$$
$$0 < r, \rho_1, \rho_2 < 1, \quad \eta_1, \eta_2 \in \mathbb{R}. \tag{3.6}$$

Note that $\phi(r, 1, \eta_1, \rho_2, \eta_2) \in M_{(0,1)_\infty}$ and $\phi(r, \rho_1, \eta_1, 1, \eta_2) \in M_{(1,0)_\infty}$.

We now look for functions $\rho_1, \rho_2, \eta_1, \eta_2 : \partial\Delta \to \mathbb{R}$ such that

$$\begin{cases} \phi\left(r, \rho_1\left(e^{i\theta}\right), \rho_2\left(e^{i\theta}\right), \eta_1\left(e^{i\theta}\right), \eta_2\left(e^{i\theta}\right)\right) \text{ extends holomorphically to } \Delta, \\ \text{for each } \theta, \rho_1\left(e^{i\theta}\right) = 1 \text{ or } \rho_2(e^{i\theta}) = 1. \end{cases} \tag{3.7}$$

To satisfy the first condition of (3.7), the function $\rho_1(e^{i\theta})e^{i\eta_1(e^{i\theta})}$ has to extend holomorphically. This happens if and only if

$$\eta_1(e^{i\theta}) = T_1 \log\left(\rho_1\left(e^{i\theta}\right)\right) + \psi_1. \tag{3.8}$$

Analogously, looking at the second component, we obtain the condition

$$\eta_2(e^{i\theta}) = T_1 \log\left(\rho_2\left(e^{i\theta}\right)\right) + \psi_2. \tag{3.9}$$

Here ψ_1 and ψ_2 are constants and T_1 is the Hilbert transform on the unit disc normalized by the condition $T_1 u(1) = 0$. Note that the third component of ϕ is automatically holomorphic when the first two components are holomorphic and never zero. Let now $p = (p_1, p_2)$, and let $t \in \left[\frac{1}{|p|^2}, \frac{1}{|p|}\right)$. Then $tp = (tp_1, tp_2) \in \mathbb{B}^2$. Lemma 2.3 implies that the point $Q_t = \left((tp_1, tp_2); \left[\overline{p}\left(t + 1 - 2t^2|p|^2\right)\right]\right)$ is the only point in M_p such that $\pi(Q_t) = tp$, where π is the natural projection from the cotangent bundle. We now look for a family of analytic discs $\{B_t\}$ (see Fig. 2) such that

$$\begin{cases} B_t(\partial\Delta) \subset M^{reg}_{(1,0)_\infty} \cup M^{reg}_{(0,1)_\infty} \\ B_t(0) = Q_t \quad \forall\, t. \end{cases} \tag{3.10}$$

Let ρ_j, η_j for $j = 1, 2$ be solutions of (3.8) and (3.9). We want to determine conditions on the ρ_j and η_j such that (3.10) holds. Looking at the second equation in (3.10), we let $\alpha_j := \widetilde{\rho_j e^{i\eta_j}}(0)$ for $j = 1, 2$, where $\widetilde{\ }$ denotes the holomorphic extension to the unit disc. The α_j must satisfy:

$$\begin{cases} r\alpha_1 = tp_1 \\ \sqrt{1-r^2}\alpha_2 = tp_2 \\ \frac{r\alpha_2}{\sqrt{1-r^2}\alpha_1} = \frac{\overline{p_1}}{\overline{p_2}}. \end{cases} \tag{3.11}$$

The solution to (3.11) is given by

$$\begin{cases} r = \frac{|p_1|}{|p|} \\ \alpha_1 = t|p|\frac{p_1}{|p_1|} \\ \alpha_2 = t|p|\frac{p_2}{|p_2|}. \end{cases}$$

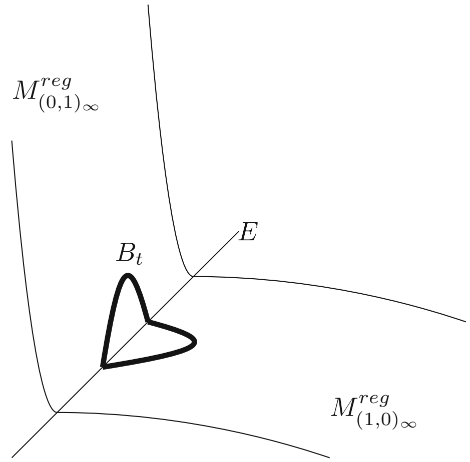


Fig. 2 An analytic disc B_t attached to the wedge whose faces are $M^{reg}_{(1,0)_\infty}$ and $M^{reg}_{(0,1)_\infty}$

Let $\left\{\rho^t_j : \partial\Delta \to (0,1],\ t \in \left[\frac{1}{|p|^2}, \frac{1}{|p|}\right)\right\}$ for $j = 1, 2$ be two continuous families of smooth functions such that, for all t, the following conditions are satisfied:

$$\begin{cases} \rho^t_1(e^{i\theta}) = 1 \text{ for } 0 \le \theta \le \pi \\ \rho^t_2(e^{i\theta}) = 1 \text{ for } \pi \le \theta \le 2\pi \\ \frac{1}{2\pi}\int_0^{2\pi} \log(\rho^t_j(e^{i\theta}))d\theta = \log(t|p|)\ j = 1, 2. \end{cases}$$

Moreover, for each t, let ψ^t_j for $j = 1, 2$ be constants such that

$$\psi^t_j + \frac{1}{2\pi}\int_0^{2\pi} T_1 \log\left(\rho^t_j\left(e^{i\theta}\right)\right) d\theta = \arg(p_j).$$

Now choose two families of functions $\left\{\eta^t_j,\ t \in \left[\frac{1}{|p|^2}, \frac{1}{|p|}\right)\right\}$ such that (3.8) and (3.9) are both satisfied for each t, with $\rho_1, \rho_2, \psi_1, \psi_2$ replaced by $\rho^t_1, \rho^t_2, \psi^t_1, \psi^t_2$. We thus obtain the family of discs $B_t = \phi\left(r, \rho^t_1, \eta^t_1, \rho^t_2, \eta^t_2\right)$ satisfying (3.10).

Note that, for $t \to \frac{1}{|p|}$, the disc B_t shrinks to the point $\left(p, \frac{\overline{p_1}}{p_2}\right)$. Therefore, for values of t close to $\frac{1}{|p|}$, the disc B_t is contained in the neighborhood of E where F is holomorphic. □

Remark 3.1 The hypothesis of orthogonality of the testing families was used in the construction of the discs $\{B_t\}$. With that assumption, it was possible to attach the discs to $M^{reg}_{(1,0)_\infty} \cup M^{reg}_{(0,1)_\infty}$ by elementary techniques.

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