Type conditions for real hypersurfaces in $\mathbb{C}^n$

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Proposition 1 (Holomorphic Decomposition)

If M is real analytic, then locally at 0

$$r = 2\,\mathrm{Re}(h) + \sum_{j=1}^{\infty} |f_j|^2 - \sum_{j=1}^{\infty} |g_j|^2,$$

where $h, f_j, g_j \in \mathcal{O}_n$.

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(In the case (1) the ideal I is replaced by a family of ideals).

Example in $\mathbb{C}^3$

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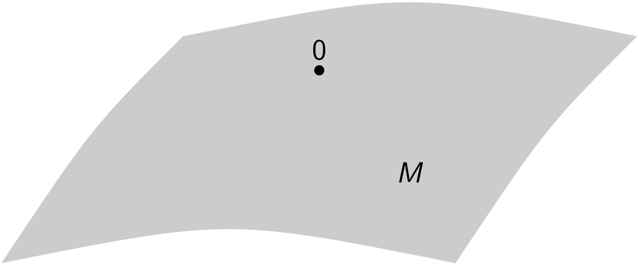
Does the hypersurface M contain the germ of a (non-trivial) analytic set of dimension q at 0?

The case $q = 1$

Does the hypersurface M contain a germ of a non-constant holomorphic curve at 0?



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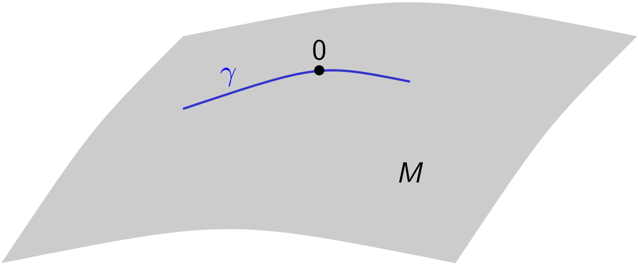

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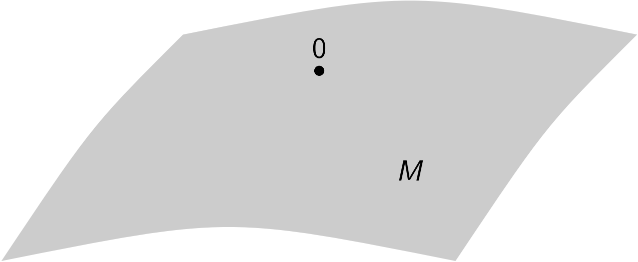

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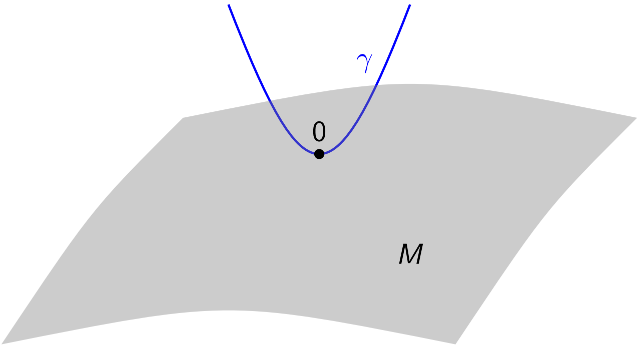
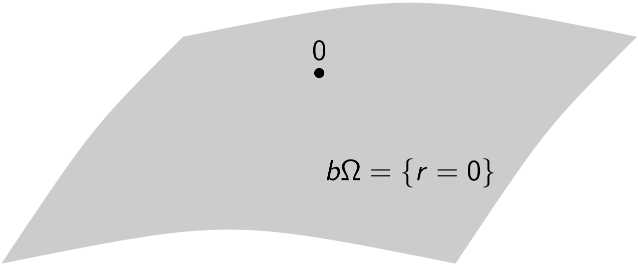


In this case, what is germ of a complex curve γ that "touches" M to highest order?

Complex domains

Let $\Omega \subset \mathbb{C}^n$ be a domain with smooth boundary. Locally



$\Omega = \{r < 0\}$

Balls and Polydiscs

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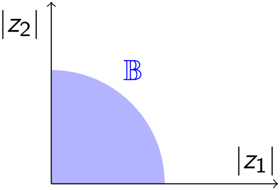

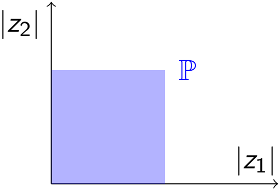

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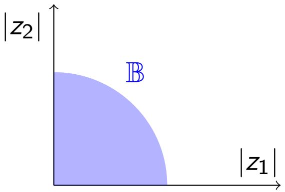

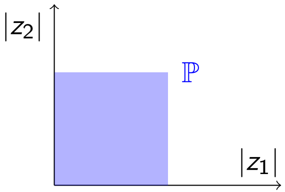

They are not biholomorphically equivalent, since $\mathbb{B}$, unlike $\mathbb{P}$, has no analytic structure in its boundary.

Definition of the 1-type

$M \subset \mathbb{C}^n$ smooth real hypersurface defined at 0 by $r = 0$.

$$\mathbf{T}_1(M) := \sup_{\gamma \in \Gamma} \frac{v(r \circ \gamma)}{v(\gamma)}$$

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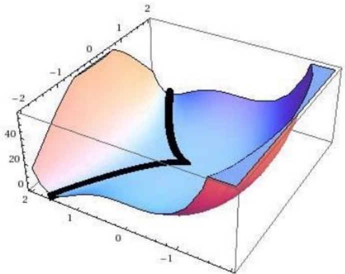

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For all hypersurfaces in our special class:

$$\mathbf{T}_1(M) = 2\mathbf{T}_1(I)$$

Another Example

$$I = (z_1^2 - z_2^5, z_1^3 z_2) \subset \mathcal{O}_2$$

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giving respectively $2, 5$ and $17/2$.

Therefore $\mathbf{T}_1(I) = 17/2$.

Curve selection

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Theorem 2 (McNeal and Nemethi, 2005)

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(Heier and Lazarsfeld, 2008) generalized this result to $\mathcal{O}_n$.

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We define for $1 < q \leq n$

$$\mathbf{T}_q(I) := \inf_{w_1,\ldots,w_{q-1}} \mathbf{T}_1(I, w_1, \ldots, w_{q-1}),$$

where the w_j are linear functions.

Relation with the $\overline{\partial}$-problem

Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary $b\Omega$.

The $\overline{\partial}$-problem

Given a closed $(p, q+1)$ form on Ω, find a (p, q) form u such that

$$\overline{\partial} u = \alpha \text{ in } \Omega. \tag{2}$$

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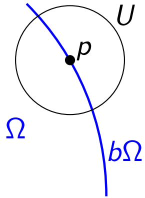
Local regularity at $p \in b\Omega$



$$\alpha \in C^\infty(U \cap \overline{\Omega}) \Longrightarrow \exists u \in C^\infty(U \cap \overline{\Omega}) \text{ solving (3)}$$

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Theorem 3 (Kohn-Nirenberg, 1965)

If a subelliptic estimate holds at p on $(0, q)$ *forms for some* $\epsilon \in (0, 1)$*, then local regularity holds at p for* $(0, q)$ *forms.*

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If $b\Omega$ *is real analytic, then the following are equivalent:*

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Relation with the $\overline{\partial}$-problem

Theorem 5 (Catlin, 1983 and 1987)

If $b\Omega$ is smooth, then the following are equivalent:

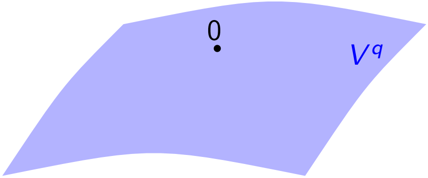
- *A subelliptic estimate holds at p on $(0, q)$ forms for some $\epsilon \in (0, 1)$.*
- *The order of contact at p with q-dimensional analytic varieties (q-type) is finite.*

Moreover, if τ is the q-type at p, then $\epsilon > \tau^{-n^2 \tau^{n^2}}$.

Relation with the $\overline{\partial}$-problem

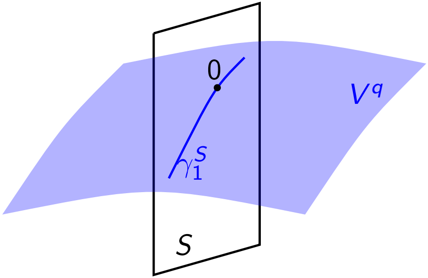
Unfortunately, Catlin has a different way of measuring the order of contact!

Fix a q-dimensional variety V^q through 0.



Catlin's q-type of an ideal $I \subset \mathcal{O}_n$

Fix a q-dimensional variety V^q through 0. Take a $(n-q+1)$-plane S and consider $V^q \cap S = \{\gamma_1^S, \ldots, \gamma_k^S\}$.



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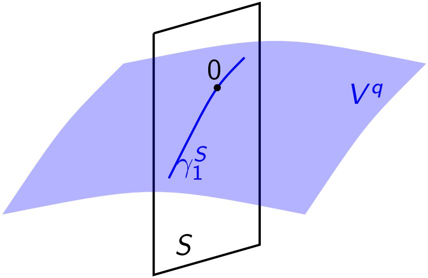

Proposition 6 (Catlin)

Fix $g \in I$. For a generic choice of S the number k is constant and the following quantity is the same:

$$\max_{j=1,\ldots,k} \frac{v(g \circ \gamma_j^S)}{v(\gamma_j^S)}.$$

Catlin's q-type

We thus have, for every $g \in I$ and V^q

$$\operatorname*{gen.\,val}_{S \in G^{n-q+1}} \max_{j=1,\dots,k} \frac{v(g \circ \gamma_j^S)}{v(\gamma_j^S)}$$

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We thus define

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Catlin's q-type

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$$\operatorname*{gen.\,val}_{S \in G^{n-q+1}} \max_{j=1,\dots,k} \frac{v(g \circ \gamma_j^S)}{v(\gamma_j^S)}$$

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$$D_q(I) := \sup_{V^q} \inf_{g \in I} \operatorname*{gen.\,val}_{S \in G^{n-q+1}} \max_{j=1,\dots,k} \frac{v(g \circ \gamma_j^S)}{v(\gamma_j^S)}$$

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$$D_1(I) = \sup_{\gamma \in \Gamma} \inf_{g \in I} \frac{v(g \circ \gamma)}{v(\gamma)} = \mathbf{T}_1(I)$$

$\mathbf{T}_q = D_q$?

In the words of Catlin

to z_0. (The definition of $D_q(z_0)$ is closely related to that of $\Delta_q(z_0)$, as defined by D'Angelo [5]. When $q = 1$, both of these numbers equal $T_1(z_0)$. The analog of this statement when $q > 1$ is probably also true.) We can now state the main

Example where $\mathbf{T}_2 < D_2$

$$I = (z_1^3 - z_3 z_2, z_2^2) \subset \mathcal{O}_3$$

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Hence $D_2(I) \geq 4$.

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Hence $D_2(I) \geq 4$. One can actually show that $D_2(I) = 4$.

Infimum vs Generic value

In the previous example

$$\mathbf{T}_2(I) = 3, \quad D_2(I) = 4$$

Recall that

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It is natural to define a new measurement

$$\beta_q(I) := \operatorname*{gen.val}_{w_1,\ldots,w_{q-1}} \ \mathbf{T}_1(I, w_1, \ldots, w_{q-1})$$

The relation between $\mathbf{T}_q$ and D_q

Theorem 7 (F., 2017)

For every $I \subset \mathcal{O}_n$ and $1 \leq q \leq n$ we have

$$\beta_q(I) = D_q(I)$$

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For every $I \subset \mathcal{O}_n$ and $1 \leq q \leq n$ we have

$$\mathbf{T}_q(I) \leq D_q(I) \leq (\mathbf{T}_q(I))^{n-q+1}$$

Proof of the Theorem

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Proving $\beta_q(I) \leq D_q(I)$ is more difficult and requires constructing an appropriate variety V^q.

Proof of $\beta_q(I) \leq D_q(I)$: the case $q = n-1$

$$\beta := \beta_{n-1}(I) = \underset{w_1,\ldots,w_{n-2}}{\text{gen. val}}\ \mathbf{T}_1(I, w_1, \ldots, w_{n-2})$$

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$$\beta = \underset{a,b,c,\dots}{\text{gen. val}}\ \mathbf{T}_1\Big(f_1(z_1, z_2, az_1 + bz_2, cz_1 + dz_2, \dots), \\ f_2(z_1, z_2, az_1 + bz_2, cz_1 + dz_2, \dots), \\ \dots, \\ f_m(z_1, z_2, az_1 + bz_2, cz_1 + dz_2, \dots)\Big)$$

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Hence $D_{n-1}(I) \geq \beta$.

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Exploits the Curve selection Theorem of Heier and Lazarsfeld to build the appropriate variety.

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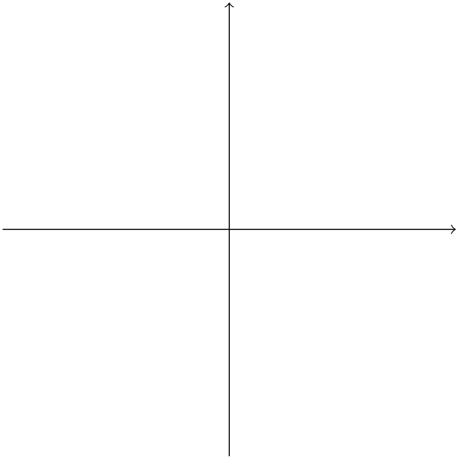
BUT we have an ideal depending on parameters!!

Proof of $\beta_q(I) \leq D_q(I)$ for general q



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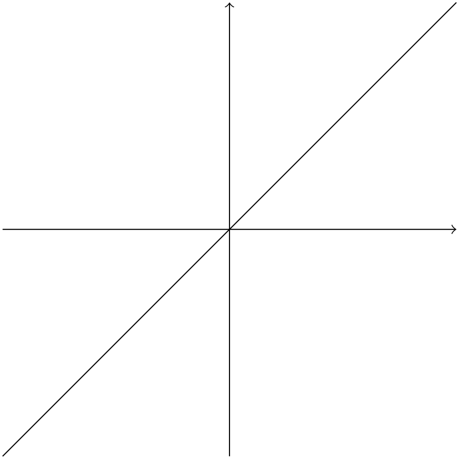

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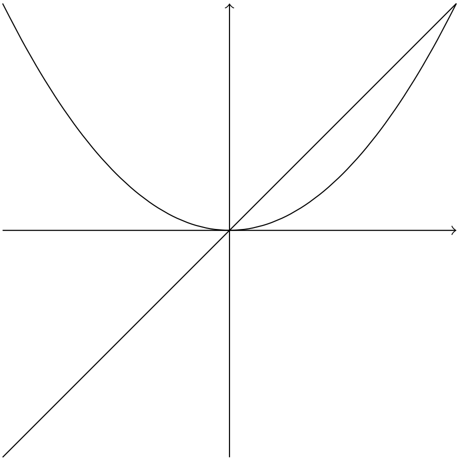

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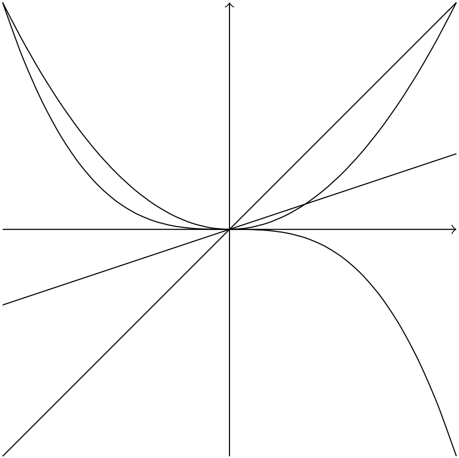

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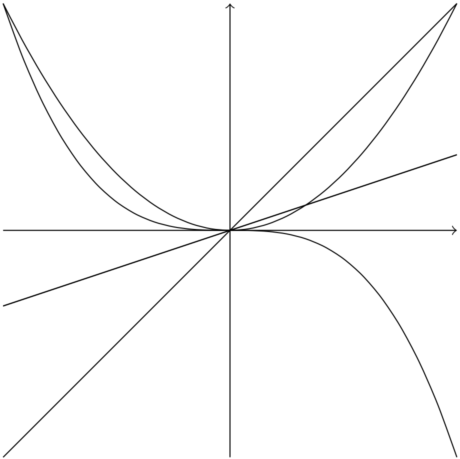

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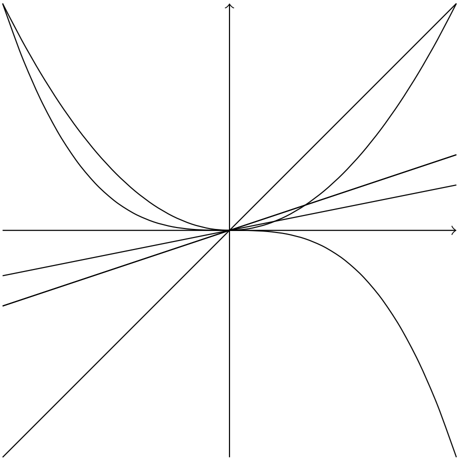

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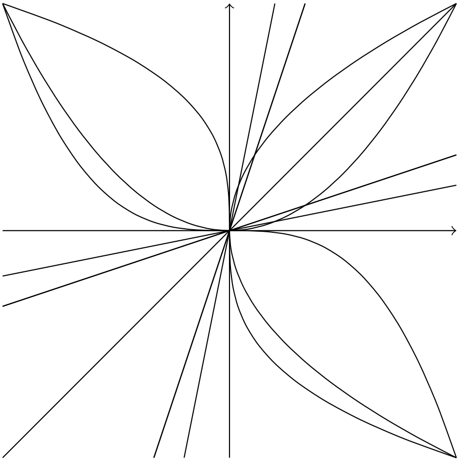

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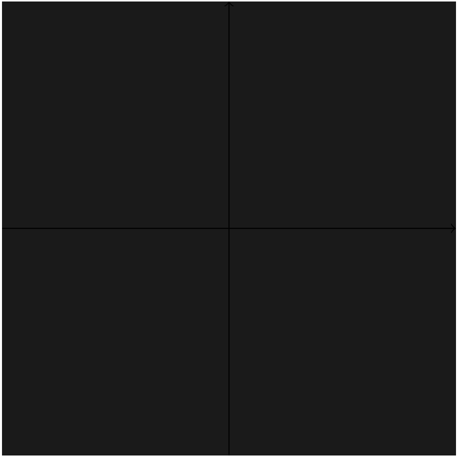

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One has to go through the proof of Heier and Lazarsfeld to check that there exists a choice of a single curve with maximum order of contact for the ideal $I_{a,b,c,\ldots}$ for a generic choice of the parameters.

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One has to go through the proof of Heier and Lazarsfeld to check that there exists a choice of a single curve with maximum order of contact for the ideal $I_{a,b,c,\dots}$ for a generic choice of the parameters.

The proof exploits the very special dependence on parameters that we have in the problem.

Back to hypersurfaces

We studied the local geometry at 0 of a hypersurface defined by

$$\mathrm{Re}(z_{n+1}) + |f_1|^2 + \cdots + |f_m|^2 = 0$$

considering the ideal

$$I = (f_1, \ldots, f_m) \subset \mathcal{O}_n$$

In this case we had

$$\mathbf{T}_q(M) = 2\mathbf{T}_q(I)$$

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For the case

$$2\,\mathrm{Re}(h) + \sum_{j=1}^{m} |f_j|^2 - \sum_{j=1}^{k} |g_j|^2 = 0$$

one has to consider a family of ideals.

$\mathbf{T}_q(M)$ and $D_q(M)$

Corollary 9

For every germ of a smooth real hypersurface $M \subset \mathbb{C}^n$ at 0 and $1 \le q \le n$, we have

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Theorem 10 (Catlin, 1983 and 1987)

If $b\Omega$ *is smooth, then the following are equivalent:*

- *A subelliptic estimate holds at* p *on* $(0,q)$ *forms for some* $\epsilon \in (0,1)$.
- $\mathbf{T}_q(M,p) < \infty$.

Moreover, if $\tau = \mathbf{T}_q(M,p)$, *then* $\epsilon > \tau^{-n^2(n-q+1)\tau^{n^2(n-q+1)}}$.

Where these results can be found

- A remark on two notions of order of contact, *The Journal of Geometric Analysis*
- Type conditions for real hypersurfaces in $\mathbb{C}^n$, *in preparation*

Thank you for your attention and hospitality!