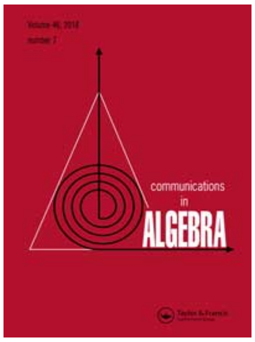

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Factorization of elements in noncommutative rings, II

Alberto Facchini[a] and Martino Fassina[b]

[a]Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova, Padova, Italy; [b]Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, Illinois, USA

ABSTRACT
The mathematical object that describes the factorizations of an element a of an arbitrary ring R is the partially ordered set of all principal right ideals of R that contain aR. We present a hierarchy of situations that depend on the structure of this partially ordered set and describe the factorizations of a.

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1. Introduction

In this paper, we study factorizations of elements in arbitrary rings with identity. The classical theory of factorization concerns factorizations of elements of a commutative integral domain and deals with the concepts of irreducible element, prime element and unique factorization domain. Various studies in the mathematical literature have tried to extend this theory of unique factorization to the case of commutative rings with zero divisors and to non-commutative rings. The best results in the study of factorizations in the non-commutative case are those obtained by Cohn from 1962 on, which led to the theory of firs and are in large part presented in his book "Free rings and their relations" [7]. In this paper, we partially follow his direction.

The main idea underlying our work is the following. The mathematical object that describes the factorizations of an element a of an arbitrary ring R is the partially ordered set of all principal right ideals of R that contain aR: the better the structure of this partially ordered set, the better the behavior of the factorizations of a. Thus, in this paper, we study a hierarchy of situations that depend on the structure of this partially ordered set. We stress that our point of view is relative to the element a, it is a "local" point of view. This is in contrast to the "global" point of view, which is usually adopted in the study of factorizations of all non-zero elements of an integral domain and in the study of UFDs, possibly non-commutative, where one tries to write every non-zero element of the integral domain as a product of irreducible elements. As a consequence of this change of perspective, some of the main ingredients of the theory require new definitions. For instance, we replace the concept of factorization of an element a into irreducibles with the concept of maximal factorization of the element a, which can exist even for $a = 0$.

We therefore present our hierarchy of situations, the best results being obtained for the class of right Bézout rings. We point out that several interesting classes of right Bézout rings have been studied recently. This is the case, for instance, of the ring of entire functions (functions holomorphic on the whole complex plane), the ring of all algebraic integers and the Leavitt path algebra of any directed graph with coefficients in a field K, which are left and right Bézout rings [1].

CONTACT Martino Fassina fassina2@illinois.edu Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA.

This article is partially a continuation of our previous paper [10], where we considered factorizations of right regular elements, but the two papers can be read independently.

The rings we deal with are associative rings R, not necessarily commutative, with an identity $1 \neq 0$. If a is an element of R, its *right annihilator* $\mathrm{r.ann}_R(a)$ is the set of all $r \in R$ with $ar = 0$.

2. The hierarchy

Let R be a ring with identity. In the study of the factorizations of the elements of R, Theorem 6 in [6] suggests we proceed as follows. For any ring R, the modular lattice $\mathcal{L}(R_R)$ of all right ideals of R has as a subset the set $\mathcal{L}_p(R_R) := \{ aR \mid a \in R \}$ of all principal right ideals of R. The lattice structure on $\mathcal{L}(R_R)$ induces a partial order on $\mathcal{L}_p(R_R)$. Thus we have a mapping $\varphi \colon R \to \mathcal{L}_p(R_R)$ and the inclusion $\varepsilon \colon \mathcal{L}_p(R_R) \to \mathcal{L}(R_R)$, where the mapping ε is an embedding of partially ordered sets, and the mapping φ is order-reversing when R is endowed with the preorder $|_l$. Recall that if a, b are elements of a ring R, we say that a is a *left divisor of* b (in R), and write $a|_l b$, if there exists an element $x \in R$ with $ax = b$. The factorizations of any element $a \in R$ are described by the closed interval $[aR, R]_{\mathcal{L}_p(R_R)}$ of the elements between aR and R in the partially ordered set $\mathcal{L}_p(R_R)$. Thus an element $a \in R$ is a left irreducible element if and only if $a \neq 0$ and the interval $[aR, R]_{\mathcal{L}_p(R_R)}$ has exactly two elements [10, Lemma 1].

Let R be any ring and a an arbitrary element of R. The factorizations of a are determined by the interval $[aR, R]_{\mathcal{L}_p(R_R)}$ of all principal right ideals of R between aR and R, and the best situation is when $[aR, R]_{\mathcal{L}_p(R_R)}$ turns out to be a direct product of finitely many chains of finite length. We have a variety, almost a hierarchy, of possible cases:

(a) $[aR, R]_{\mathcal{L}_p(R_R)}$ is an arbitrary partially ordered set;
(b) $[aR, R]_{\mathcal{L}_p(R_R)}$ is a graded partially ordered set;
(c) $[aR, R]_{\mathcal{L}_p(R_R)}$ is a lattice;
(d) $[aR, R]_{\mathcal{L}_p(R_R)}$ is a modular lattice;
(e) $[aR, R]_{\mathcal{L}_p(R_R)}$ is a semimodular lattice of finite length (see [13]);
(f) $[aR, R]_{\mathcal{L}_p(R_R)}$ is a sublattice of $\mathcal{L}(R_R)$;
(g) $[aR, R]_{\mathcal{L}_p(R_R)}$ is a sublattice of finite length of $\mathcal{L}(R_R)$;
(h) $[aR, R]_{\mathcal{L}_p(R_R)}$ is a sublattice $\mathcal{L}(R_R)$ and is a direct product of finitely many chains.

Recall that an element a of a ring R is a *left zerodivisor* if it is non-zero and there exists $b \in R$, $b \neq 0$ such that $ab = 0$, and is *right regular* if it is non-zero and is not a left zerodivisor.

Let us analyze these different possibilities in detail.

Case (a): $[aR, R]_{\mathcal{L}_p(R_R)}$ *is a partially ordered set (of finite length or infinite length).* We deal with finite chains in a partially ordered set. Let $(P, \leq)$ be a partially ordered set. By a *finite chain* in P we mean a finite indexed set $C = \{p_i \mid i = 1, 2, \dots, n\}$ with $p_i \in P$ for every i and $p_1 \leq p_2 \leq \cdots \leq p_n$. We say that two chains $C = \{p_i \mid i = 1, 2, \dots, n\}$ and $C' = \{q_j \mid j = 1, 2, \dots, m\}$ with $p_i, q_j \in P$, $p_1 \leq p_2 \leq \cdots \leq p_n$ and $q_1 \leq q_2 \leq \cdots \leq q_m$ are *equal* if $n = m$ and $p_i = q_i$ for every $i = 1, 2, \dots, n$. The *length* $l(C)$ of the chain C is the cardinality of the set C minus one: $l(C) = |C| - 1$. Here $|C|$ denotes the number of distinct elements p_i in C. Thus $l(C) \leq n - 1$. The partially ordered set P is said to be *of length* n, where n is a natural number, if there is a chain in P of length n and all chains in P are of length $\leq n$. A partially ordered set P is *of finite length* if it is of length n for some natural number n [12, p. 2]. Every chain in $[aR, R]_{\mathcal{L}_p(R_R)}$ corresponds to a factorization of a. Let us be more precise on this point and see how the definitions must be modified in passing from the case of right regular elements [10] to the case of arbitrary elements of a ring R. Recall that, at the beginning of Section 4 of [10], we have defined two factorizations $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_m)$ into right regular elements to be equivalent if $n = m$ and there exists $u_1, u_2, \dots, u_{n-1} \in U(R)$ such that $(b_1, b_2, \dots, b_m) = (a_1u_1, u_1^{-1}a_2u_2, u_2^{-1}a_3u_3, \dots, u_{n-1}^{-1}a_n)$. Now, for an arbitrary ring R, we say that two factorizations $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_m)$ of an element $a \in R$ are *equivalent* if $n = m$ and there exists $u_1, v_1, u_2, v_2, \dots, u_{n-1}, v_{n-1} \in R$ and $t_i \in \mathrm{r.ann}(a_1a_2 \dots a_{i-1}u_{i-1})$ for every $i = 1, 2, \dots, n$ such that $u_iv_i - 1 \in \mathrm{r.ann}_R(a_1a_2\dots a_i)$ for every $i = 1, 2, \dots, n-1$ and $(b_1, b_2, \dots, b_m) = (a_1u_1, v_1a_2u_2 + t_2, v_2a_3u_3 + t_3, \dots, v_{n-1}a_n + t_n)$. This definition is motivated by the following theorem, which is the extension of [10, Proposition 3] to the general case.

Theorem 2.1. *Let a be an element of a ring R,*

$$\mathcal{F}(a) := \{ (a_1, a_2, \dots, a_n) \mid n \geq 1,\ a_i \in R,\ a_1a_2\dots a_n = a\}$$

the set of all factorizations of a, and

$$\mathcal{C}_a := \{ (aR, I_1, I_2, \dots, I_{n-1}, R) \mid n \geq 1, I_j \text{ a principal right ideal of } R\}$$

the set of all finite chains of principal right ideals from aR to R_R. Let $f\colon \mathcal{F}(a) \to \mathcal{C}_a$ be the mapping defined by

$$f(a_1, a_2, \dots, a_n) = (aR = a_1a_2\dots a_nR, a_1a_2\dots a_{n-1}R, \dots, a_1R, R)$$

for every $(a_1, a_2, \dots, a_n) \in \mathcal{F}(a)$. Then the mapping f is surjective, and two factorizations in $\mathcal{F}(a)$ are mapped via f to the same element of $\mathcal{C}_a$ if and only if they are equivalent factorizations of a.

Proof. As far as the surjectivity of f is concerned, fix a chain $aR \subseteq b_1R \subseteq \cdots \subseteq b_{n-1}R \subseteq R$ of principal right ideals. Set $b_0 := a$ and $b_n := 1$. For every $i = 1, 2, \dots, n$, there exists $c_i \in R$ with $b_{i-1} = b_ic_i$. Then, for every $i = 1, 2, \dots, n$, $b_{n-i} = c_nc_{n-1}\dots c_{n-i+1}$, as can be easily proved by induction on i. In particular, $c_nc_{n-1}\dots c_1 = b_0 = a$ is a factorization of a, and its associated chain of principal right ideals is exactly $aR \subseteq b_1R \subseteq b_2R \subseteq \cdots \subseteq b_{n-1}R \subseteq R$. This proves that f is an onto mapping.

Now suppose that two factorizations $a = a_1\dots a_n = b_1\dots b_m$ are mapped to the same chain of principal right ideals, where $a_i, b_j \in R$, that is, $n = m$ and $a_1a_2\dots a_iR = b_1b_2\dots b_iR$ for every $i = 1, 2, \dots, n-1$. Hence, for every $i = 1, 2, \dots, n-1$, there exist $v_i, u_i \in R$ such that $a_1a_2\dots a_i = b_1b_2\dots b_iv_i$ and $a_1a_2\dots a_iu_i = b_1b_2\dots b_i$. Then $a_1a_2\dots a_i = b_1b_2\dots b_iv_i = a_1a_2\dots a_iu_iv_i$, so that $a_1a_2\dots a_i(u_iv_i - 1) = 0$. Similarly, $b_1b_2\dots b_i(v_iu_i - 1) = 0$. Then $b_1 = a_1u_1$. Set $u_n := 1$. Let us show that $b_i - v_{i-1}a_iu_i \in \mathrm{r.ann}(a_1a_2\dots a_{i-1}u_{i-1})$ for $i = 2, 3, \dots, n$.

From $b_1\dots b_iv_i = a_1\dots a_i = (a_1\dots a_{i-1})a_i = b_1b_2\dots b_{i-1}v_{i-1}a_i$ it follows that $b_1\dots b_i = b_1\dots b_iv_iu_i = b_1b_2\dots b_{i-1}v_{i-1}a_iu_i$. We therefore have, for $i = 2, 3, \dots, n$,

$$b_i - v_{i-1}a_iu_i \in \mathrm{r.ann}(b_1b_2\dots b_{i-1}) = \mathrm{r.ann}(a_1a_2\dots a_{i-1}u_{i-1}).$$

Conversely, suppose that $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n)$ are such that $a = a_1\dots a_n = b_1\dots b_n$ and there exist $u_1, v_1, u_2, v_2, \dots, u_{n-1}, v_{n-1} \in R$ and $t_i \in \mathrm{r.ann}_R(a_1a_2\dots a_{i-1}u_{i-1})$ for every $i = 2, 3, \dots, n$ with $u_iv_i - 1 \in \mathrm{r.ann}_R(a_1a_2\dots a_i)$ for every $i = 1, 2, \dots, n-1$ and

$$(b_1, b_2\dots, b_m) = (a_1u_1, v_1a_2u_2 + t_2, v_2a_3u_3 + t_3, \dots, v_{n-1}a_n + t_n).$$

We claim that $a_1a_2\dots a_iR = b_1b_2\dots b_iR$ for every $i = 1, 2, \dots, n-1$. For $i = 1$, we have $b_1R = a_1u_1R \subseteq a_1R$ and $a_1R = a_1u_1v_1R = b_1v_1R \subseteq b_1R$. For $i = 2, \dots, n-1$

$$\begin{aligned}
b_1b_2\dots b_iR &= a_1u_1(v_1a_2u_2 + t_2)(v_2a_3u_3 + t_3)\dots(v_{i-1}a_i + t_i)R \\
&= a_1a_2u_2(v_2a_3u_3 + t_3)\dots(v_{i-1}a_i + t_i)R \\
&= a_1a_2a_3\dots(v_{i-1}a_i + t_i)R \\
&= a_1a_2\dots a_{i-1}u_{i-1}(v_{i-1}a_i + t_i)R = a_1a_2\dots a_iR,
\end{aligned}$$

as wanted. □

We call *right length* of a factorization $a = a_1a_2\dots a_n$ the number of pairwise distinct right ideals in the chain $a_1a_2\dots a_nR \subseteq a_1a_2\dots a_{n-1}R \subseteq \cdots \subseteq a_1a_2R \subseteq a_1R$.

Recall that if a, b are elements of a ring R, we say that a and b are *left associates*, and write $a \sim_l b$, if $a|_l b$ and $b|_l a$ [10]. If $a, b \in R$, a is a *maximal left divisor of* b if a is a left divisor of b, the elements a and b are not left associates and, for every $c \in R$, $c|_l b$ and $a|_l c$ imply that either $c \sim_l a$ or $c \sim_l b$. Equivalently, a is a maximal left divisor of b if and only if the interval $[bR, aR]_{\mathcal{L}_p(R_R)}$ has exactly two elements, necessarily, aR and bR.

Lemma 2.2. *If $[aR, R]_{\mathcal{L}_p(R_R)}$ is a partially ordered set of finite length, then the right length of the factorizations of a is bounded and there exists a maximal left factorization of a, that is, a factorization $a = a_1 \dots a_n$ for which $a_1 a_2 \dots a_{i-1}$ is a maximal left divisor of $a_1 a_2 \dots a_i$ for every $i = 1, 2, \dots, n$. If $[aR, R]_{\mathcal{L}_p(R_R)}$ is of infinite length, then the right length of the factorizations of a is unbounded.*

Proof. If $[aR, R]_{\mathcal{L}_p(R_R)}$ is a partially ordered set of finite length n, then $[aR, R]_{\mathcal{L}_p(R_R)}$ has a maximal chain of length n, which can be written in the form $aR \subset b_{n-1}R \subset \cdots \subset b_1R \subset R$. Set $b_0 := 1$ and $b_n := a$. For every $i = 1, 2, 3, \dots, n$ there exists $a_i \in R$ with $b_i = b_{i-1}a_i$. Then $b_i = a_1 a_2 \dots a_i$ for every $i = 1, 2, 3, \dots, n$. Since the interval $[b_iR, b_{i-1}R]_{\mathcal{L}_p(R_R)}$ has exactly two elements, we have that b_{i-1} is a maximal left divisor of b_i, that is, $a_1 a_2 \dots a_{i-1}$ is a maximal left divisor of $a_1 a_2 \dots a_i$. The statement about the interval of infinite length is now trivial. □

For example, the factorizations of any right invertible element $a \in R$ have all right length 0. If $z \in \mathbb{Z}$, then $[z\mathbb{Z}, \mathbb{Z}]_{\mathcal{L}_p(\mathbb{Z}_\mathbb{Z})}$ has finite length for $z \neq 0$, and has infinite length if $z = 0$, so that 0 has factorizations in $\mathbb{Z}$ of arbitrary length. If R is a right artinian ring, so that R_R has finite composition length, then all the factorizations of 0, and of any element of R, have finite length.

This shows us how it is convenient to proceed in the case of an element a of an arbitrary ring R. It is necessary to replace the concept of left irreducible element x, which is an "absolute" concept (a non-zero element $x \in R$ is a left irreducible element if and only if the interval $[xR, R]_{\mathcal{L}_p(R_R)}$ has exactly two elements), with the "relative" concept of maximal left divisor of a (x is a maximal left divisor of a if and only if the interval $[aR, xR]_{\mathcal{L}_p(R_R)}$ has exactly two elements). Clearly, $a \in R$ is a left irreducible element if and only if 1 is a maximal left divisor of a.

Lemma 2.3. *The following conditions are equivalent for a left divisor a of an element b in a ring R:*

(1) *a is a maximal left divisor of b.*

(2) *a and b are not left associates, and for every $x, y \in R$ with $b = axy$, the right ideal axR is equal to either aR or $axyR$.*

Proof. Suppose that a is a maximal left divisor of b. Then the interval $[bR, aR]_{\mathcal{L}_p(R_R)}$ has exactly two elements. In particular, $bR \neq aR$, so a and b are not left associates. Moreover, if $x, y \in R$ are such that $b = axy$, then $bR = axyR \subseteq axR \subseteq aR$, hence axR is equal to either aR or $axyR$. Conversely, if (2) holds, then $aR \neq bR$. Moreover, let $c \in R$ be such that $bR \subseteq cR \subseteq aR$. Then there exists $x, y \in R$ such that $c = ax$ and $b = cy = axy$. By hypothesis, either $cR = aR$ or $cR = bR$. Therefore $[bR, aR]_{\mathcal{L}_p(R_R)}$ has exactly two elements, and (1) holds. □

Lemma 2.4. *Let a be a left divisor of an element $b = ar$ in a ring R and suppose that a is right regular. Then a is a maximal left divisor of b if and only if r is a left irreducible element of R.*

Proof. Suppose a is a maximal left divisor of b and let $x \in R$ be such that $rR \subseteq xR \subseteq R$. Then there exists $y \in R$ such that $r = xy$. Since $b = ar = axy$, from Lemma 2.3 we obtain that either $axR = axyR$ or $axR = aR$. Since a is right regular this implies that either $xR = xyR = rR$ or $xR = R$, hence r is left irreducible. Conversely, assume that r is left irreducible, and let $x, y \in R$ be such that $b = axy$. Since $b = ar$ and a is right regular we have that $r = xy$. From $axyR \subseteq axR \subseteq aR$ we obtain, by right regularity of a, $xyR \subseteq xR \subseteq R$. Since r is left irreducible, we have that either $xR = R$ or $xR = rR$, hence either $axR = aR$ or $axR = arR = axyR$. Therefore, a is a maximal left divisor of b by Lemma 2.3. □

Recall that a ring R is a *right saturated* ring if every left divisor of a right regular element is right regular, that is, if for every $a, b \in R$, ab is right regular if and only if both a and b are right regular [10, Section 2].

Proposition 2.5. *Let a be a right regular element of a right saturated ring R and suppose that $[aR, R]_{\mathcal{L}_p(R_R)}$ is a partially ordered set of finite length. Then a has a maximal left factorization, and every maximal left factorization $a = a_1 \dots a_n$ of a consists of left irreducible elements $a_1, \dots, a_n$ of R.*

Proof. By Lemma 2.2, there exists a maximal left factorization $a = a_1 \dots a_n$ of a. For any such maximal left factorization $a = a_1 \dots a_n$ of a, we have that $a_1 a_2 \dots a_{i-1}$ is a maximal left divisor of $a_1 a_2 \dots a_i$ for every $i = 2, 3, \dots, n$. Since $a = a_1 \dots a_n$ is right regular and R is right saturated, we have that $a_1 \dots a_i$ is right regular for every i. Thus, by Lemma 2.4, every a_i is left irreducible. □

Recall that, according to Cohn [8, p. 43], a non-zero element a of an integral domain R is *rigid* if $a = bc = b'c'$ implies $b \in b'R$ or $b' \in bR$. More generally, this definition can be extended to any element a of any ring R. We say that an element a of a ring R is *right rigid* if, for every $b, b', c, c' \in R$, $a = bc = b'c'$ implies $b \in b'R$ or $b' \in bR$. Thus, an element $a \in R$ is right rigid if and only if the principal right ideals between aR and R form a chain under inclusion. The principal right ideals between aR and R form a *finite* chain under inclusion if and only if a has a maximal left factorization unique up to equivalence of factorizations (Theorem 2.1 and Lemma 2.2).

We say that an element a of a ring R is *right hollow* if, for every right irreducible element $b \in R$, $b|_r a$ implies that b is left invertible modulo $\mathrm{r.ann}_R(a)$, that is, $Rb + \mathrm{r.ann}_R(a) = R$. For instance, 0 is right hollow in any ring R, and every left invertible element is right hollow. If R is a ring with no right irreducible elements, every element of R is right hollow.

Proposition 2.6. *Let a be an element of a ring R. If the partially ordered set $[aR, R]_{\mathcal{L}_p(R_R)}$ is noetherian, then a has a factorization $a = bc_1 \dots c_n$ as a product of a right hollow element $b \in R$ and $n \geq 0$ right irreducible elements $c_1, \dots, c_n$.*

Proof. Suppose $[aR, R]_{\mathcal{L}_p(R_R)}$ noetherian. Construct by induction two sequences $a_0, a_1, a_2, \dots$ and $b_1, b_2, b_3, \dots$ of elements of R as follows. Set $a_0 := a$. If a_n has been constructed and is not right hollow, then there exists a right irreducible element $b_{n+1} \in R$ with $Ra_n \subseteq Rb_{n+1}$ and b_{n+1} not left invertible modulo $\mathrm{r.ann}_R(a_n)$. Hence there exists $a_{n+1} \in R$ such that $a_n = a_{n+1}b_{n+1}$. This completes the construction of a_{n+1} and b_{n+1} for every integer $n \geq 0$ with a_n not right hollow. Now we have two cases: either a_{n_0} is right hollow for some n_0, so that the two sequences of elements a_i, b_i end at the n_0-th step, or a_n is not right hollow for every n.

We claim that a_{n_0} is right hollow for some n_0. To prove the claim, suppose the contrary, i. e., that for every n the element a_n is not right hollow. Notice that $a = a_0 = a_1 b_1 = a_2 b_2 b_1 = \dots = a_n b_n b_{n-1} \dots b_1$. Thus $aR \subseteq a_n R = a_{n+1} b_{n+1} R \subseteq a_{n+1} R$. Since $[aR, R]_{\mathcal{L}_p(R_R)}$ is noetherian, we get that $a_{n_0} R = a_{n_0+1} R$ for a suitable n_0, so that $a_{n_0+1} = a_{n_0} r$ for some $r \in R$. Then $a_{n_0} = a_{n_0+1} b_{n_0+1} = a_{n_0} r b_{n_0+1}$. Hence b_{n_0+1} is left invertible modulo $\mathrm{r.ann}_R(a_{n_0})$, which contradicts the way in which we have constructed the b_i's. This contradiction proves the claim, so that the construction ends at the n_0-th step, and $a = a_n b_n b_{n_1} \dots b_1$ turns out to be the required factorization of a. □

Lemma 2.7. *Let a be an element of an integral domain R. Then:*
(1) *a is right invertible if and only if it is left invertible.*
(2) *a is right irreducible if and only if it is left irreducible.*

Proof.
(1) follows from [10, Lemma 2] and (2) follows from [10, Lemma 6]. □

Proposition 2.8. *If $b \neq 0$ is a right hollow element of an integral domain R, then there is no irreducible element $d \in R$ such that $d|_r b$.*

Proof. Suppose that $b \neq 0$ is a right hollow element of an integral domain R and that $d \in R$ is an irreducible element with $d|_r b$. Since b is right hollow, d is left invertible modulo $\mathrm{r.ann}_R(b)$. Now, R is an integral domain and $b \neq 0$, hence d is left invertible, a contradiction. □

Corollary 2.9. *If b is a right hollow element of an integral domain R and $[Rb, {}_RR]_{\mathcal{L}_p({}_RR)}$ is a noetherian partially ordered set, then either $b = 0$ or b is invertible.*

Proof. Suppose that b is a right hollow element of an integral domain R, $[Rb, {}_RR]_{\mathcal{L}_p({}_RR)}$ is a noetherian partially ordered set, $b \neq 0$ and b is not invertible. As R is an integral domain, b is not left invertible. Since $[Rb, {}_RR]_{\mathcal{L}_p({}_RR)}$ is a noetherian partially ordered set and $Rb \neq R$, there exists an irreducible element $d \in R$ with $Rb \subseteq Rd$. Now apply Proposition 2.8. □

Corollary 2.10. *Let a be an element of an integral domain R. If the partially ordered sets $[aR, R]_{\mathcal{L}_p(R_R)}$ and $\mathcal{L}_p({}_RR)$ are noetherian, then either $a = 0$, or a is invertible, or a has a factorization $a = c_1 \dots c_n$ as a product of finitely many irreducible elements $c_1, \dots, c_n$.*

Proof. Let a be an element of an integral domain R with $[aR, R]_{\mathcal{L}_p(R_R)}$ and $\mathcal{L}_p({}_RR)$ both noetherian. By Proposition 2.6, we get that a has a factorization $a = bc_1 \dots c_n$ as a product of a right hollow element $b \in R$ and $n \geq 0$ irreducible elements $c_1, \dots, c_n$. By Corollary 2.9, either $b = 0$ or b is left invertible. If $b = 0$, then $a = 0$ and we are done. Hence we can assume b left invertible, so that b is invertible because R is an integral domain. If $n = 0$, then $a = b$ is invertible and we are done. To conclude the case $n \geq 1$, it remains to notice that the product bc_1 of an invertible element b and a right irreducible element c_1 is right irreducible, which is easily seen. □

Example 2.11. Consider the case of a right noetherian right chain ring R. This simply means that R is a ring for which $\mathcal{L}(R_R)$ is a linearly ordered set with the ascending chain condition, i. e., that $\mathcal{L}(R_R)$ is a linearly ordered set antiisomorphic to a non-limit ordinal number $\tau = \tau' + 1$. Every right ideal of R is principal, so that $\mathcal{L}_p(R_R) = \mathcal{L}(R_R)$. Examples of such rings for any non-limit ordinal τ have been constructed by Jategaonkar [17].

Let R be a right noetherian right chain ring. Then all right ideals of R are two-sided [2, Lemma 3.2]. Thus $\mathcal{L}(R_R) \subseteq \mathcal{L}({}_RR)$, and we will see below that $\mathcal{L}(R_R) \subset \mathcal{L}({}_RR)$ and $\mathcal{L}({}_RR)$ is not linearly ordered whenever $\tau > \omega + 1$, where ω denotes the first infinite ordinal. Consider the chain of prime two-sided (=right) ideals of R:

$$P_0 \supset P_1 \supset \dots \supset P_\alpha \supset P_{\alpha+1} \supset \dots$$

Let p_α be a generator of the right ideal P_α, so that $p_\alpha R = P_\alpha$. If $p_\alpha \neq 0$, then $p_\alpha R \supset p_\alpha^2 R$ (otherwise $p_\alpha R \subseteq p_\alpha^2 R$, so that there exists $r \in R$ with $p_\alpha = p_\alpha^2 r$, hence $p_\alpha(1 - p_\alpha r) = 0$; as $p_\alpha \in P_0$, which is the Jacobson radical of R, it follows that $1 - p_\alpha r$ is invertible in R, so $p_\alpha = 0$, a contradiction). We get that $\bigcap_{n\geq 1} p_\alpha^n R = P_{\alpha+1}$ whenever p_α is not nilpotent, and $P_\beta = \bigcap_{\alpha<\beta} P_\alpha$ for every limit ordinal β [9, Lemma 5.1].

As far as two prime ideals $p_\alpha R \supset p_\beta R$ of R are concerned, that is, with $\alpha < \beta$, one has that $p_\alpha p_\beta = p_\beta u$ for some invertible element $u \in R$ [2, Lemma 3.3].

If $a \in R$ and $a \neq 0$, then there exists a minimal prime ideal $p_\alpha R$ with $p_\alpha R \supseteq aR$. Since $\bigcap_{n\geq 1} p_\alpha^n R$ is either $p_{\alpha+1}R$ or 0, we get that there exists a greatest $n \geq 1$ with $p_\alpha^n R \supseteq aR$. Thus $a = p_\alpha^n a_1$ for some $a_1 \in R$ with $a_1 R \supset p_\alpha R$. Since $\mathcal{L}(R_R)$ is a linearly ordered set antiisomorphic to a well-ordered set, we can argue by induction, getting that every right ideal aR of R ($aR \neq R$ and $aR \neq 0$) can be written in the form $aR = p_{\alpha_1}^{n_1} p_{\alpha_2}^{n_2} \dots p_{\alpha_s}^{n_s} R$ with the ordinals $\alpha_1 > \alpha_2 > \cdots > \alpha_s$ and the positive integers $n_1, n_2, \dots, n_s$ uniquely determined [2, Lemma 3.4].

This clearly recalls the well-known fact that any non-zero ordinal γ can be written uniquely in the form

$$\gamma = \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_s} n_s,$$

where $n_1, \dots, n_s$ are non-zero finite ordinals and $\alpha_1 > \cdots > \alpha_s$, which suggests the existence of a bijection that behaves well. In fact:

Theorem 2.12 ([2, Theorem 3.5]). *Let R be a right noetherian right chain ring. In the notation above, the order-reversing bijection $\mathcal{L}(R_R) \to [0, \tau[$ onto the interval $[0, \tau[$ of all ordinal numbers $< \tau$ is a monoid isomorphism when $\mathcal{L}(R_R)$ is endowed with the multiplication of right ideals and $[0, \tau[$ with the addition defined by $\alpha + \beta = \min\{\alpha + \beta, \tau'\}$ for every $\alpha, \beta \in [0, \tau[$ (that is, the minimum between τ' and the sum of the two ordinals α and β). Here $\tau = \tau' + 1$, so that τ' is the greatest element of $[0, \tau[$.*

Now, if $(a_1, a_2, \dots, a_n) \in \mathcal{F}(a)$ is a factorization of a, that is, if $a = a_1 a_2 \dots a_n$, then $aR = a_1 a_2 \dots a_n R = (a_1 R)(a_2 R) \dots (a_n R)$ because $Ra_i \subseteq a_i R$, so that we get a factorization of aR in the monoid $\mathcal{L}(R_R)$, which behaves like a sum decomposition $\gamma = \gamma_1 + \cdots + \gamma_n$ of an element γ of the monoid $[0, \tau[$.

Proposition 2.13. *Let R be a local ring and suppose that its maximal right ideal $J(R) = p_0 R$ is principal as a right ideal. The following conditions are equivalent for an element $a \in R$.*
(1) *a is left irreducible.*
(2) *a is right irreducible.*
(3) *$a \in J(R) \setminus J(R)^2$.*
(4) *$a \neq 0$ and $a = p_0 u$ for some invertible element $u \in R$.*

Proof. The proposition is trivial if R is a division ring. Thus we can suppose R not a division ring. Let p_0 denote a generator of $J(R)$ as a right ideal, i. e., suppose $J(R) = p_0 R$. The set $\mathcal{L}_p(R_R) \setminus \{0, R\}$ has $J(R) = p_0 R \neq 0$ as its greatest element. Left multiplication by p_0 is an epimorphism of R_R onto $J(R)$, so that $p_0^2 R = p_0 J(R) = p_0 R J(R) = J(R)^2$ is the greatest right ideal of R properly contained in $J(R)$. We are now ready to show that the four conditions are equivalent.

(1) $\Rightarrow$ (3) Suppose a left irreducible. By [10, Lemma 1], the right ideal aR is non-zero and is a maximal element in the set $\mathcal{L}_p(R_R) \setminus \{0, R\}$ of all non-zero proper principal right ideals of R. Thus aR is a maximal element in the set $\mathcal{L}_p(R_R) \setminus \{0, R\}$ if and only if $aR = p_0 R$, which is equivalent to (3).

(3) $\Rightarrow$ (2) Suppose (3) holds. By [10, Lemma 1], we must show that the left ideal Ra is non-zero and is a maximal element in the set $\mathcal{L}_p({}_R R) \setminus \{0, R\}$ of all non-zero proper principal left ideals of R. Now, $a \in J(R) \setminus J(R)^2$ implies $a \neq 0$, so that $Ra \neq 0$. Suppose $Ra \subseteq Rb \subset R$. Then $a = rb$ for some $r \in R$. If r is left invertible, then $Ra = Rrb = Rb$. If r is not left invertible, then $r \in J(R)$, but $b \in J(R)$, so that $a = rb \in J(R)^2$, a contradiction. Thus Ra is a maximal element in the set $\mathcal{L}_p({}_R R) \setminus \{0, R\}$ of all non-zero proper principal left ideals of R.

(2) $\Rightarrow$ (1) Assume a right irreducible. Thus the left ideal Ra is non-zero and is a maximal element in the set $\mathcal{L}_p({}_R R) \setminus \{0, R\}$ of non-zero proper principal left ideals of R. In particular, $a \neq 0$ and Ra is proper, so that $a \in J(R)$. Let us prove that $a \notin J(R)^2$. Assume the contrary. Then $a \in J(R)^2 = p_0^2 R$, so that $a = p_0^2 r$ for some $r \in R$. But then $Ra = Rp_0^2 r \subseteq J(R) \cdot Rp_0 r \subset Rp_0 r$. Since Ra is maximal in $\mathcal{L}_p({}_R R) \setminus \{0, R\}$, it follows that $Rp_0 r = R$, which is a contradiction because $p_0 R \subseteq J(R)$. This proves

that $a \in J(R)\backslash J(R)^2$, so that $aR = J(R) = p_0R$. This proves that the right ideal aR is non-zero and is a maximal element in the set $\mathcal{L}_p(R_R)\backslash\{0, R\}$ of all non-zero proper principal right ideals of R. Therefore, a is left irreducible by [10, Lemma 1].

(4) $\Rightarrow$ (1) If (4) holds, $aR = p_0R$ is a maximal right ideal, so that a is left irreducible.

(3) $\Rightarrow$ (4) Suppose $a \in J(R)\backslash J(R)^2$. Then $a \neq 0$ and $a = p_0r$ for some $r \in R$. But $r \notin J(R)$, otherwise $a \in J(R)^2$. Thus r is invertible. □

Example 2.14. Proposition 2.13 can be applied to the following ring. Let k be a field and α be an endomorphism of k that is not onto. Let $R := k_\ell[[x, \alpha]]$ be the left skew series ring, whose elements are the series $f = \sum_{i\geq 0} a_ix^i$ and where multiplication is defined by $x^ia = \alpha^i(a)x^i$ [21, 1.4]. In this ring, an element f is invertible if and only if its constant term a_0 is $\neq 0$, so that R is a local ring and its maximal ideal $J(R)$ consists of the series f with constant term $a_0 = 0$. Thus $J(R) = Rx$. If k' denotes the image of α, so that k' is a proper subfield of k, then xR is properly contained in Rx because it consists of the series $\sum_{i\geq 0} a_ix^i$ with $a_0 = 0$ and $a_i \in k'$ for every $i \geq 1$. For this ring R, we have that $J(R)^2 = Rx^2$, so that a series $f \in R$ is right irreducible in R if and only if it is left irreducible in R, if and only if $a_0 = 0$ and $a_1 \neq 0$.

Proposition 2.15. *Let R be a right noetherian right chain ring and let $I_0 = R \supset I_1 = J(R) \supset I_2 \supset \cdots \supset I_\lambda \supset \dots$ be the well-ordered chain of its right ideals. A non-zero element $a \in R$ is right hollow if and only if aR does not have an immediate predecessor, that is, if and only if $aR = I_\lambda$ for $\lambda = 0$ or for some limit ordinal λ.*

Proof. Suppose that $aR = I_\lambda$ does not have an immediate predecessor. If $\lambda = 0$, then a is invertible in the local ring R, so that a is right hollow. If $aR = I_\lambda$ for a limit ordinal λ, the element a has a factorization $a = bc_1 \dots c_n$ as a product of a right hollow element $b \in R$ and $n \geq 0$ right irreducible elements $c_1, \dots, c_n$ (Proposition 2.6). Suppose $n \geq 1$. Then $aR = bc_1 \dots c_nR = bc_1 \dots c_{n-1}J(R)$ has $bc_1 \dots c_{n-1}R$ as its immediate predecessor, a contradiction. Thus $n = 0$, so that $a = b$ is right hollow.

Conversely, suppose that aR has an immediate predecessor, so that $aR = bJ(R)$ for some element $b \in R$, i. e., bR is the immediate predecessor of aR. Then $a = br$ for some $r \in J(R) = p_0R$. Thus $a = bp_0t$ for some $t \in R$. If t is not invertible, then $t \in J(R)$, so that $a \in bJ(R)^2$, hence $aR \subseteq bJ(R)^2 \subseteq bJ(R) \subset bR$, and so bR is not the immediate predecessor of aR, which is a contradiction. The contradiction shows that $t \in R$ is invertible. Right multiplication by t is therefore an automorphism of ${}_RR$, so that p_0 right irreducible (i. e., Rp_0 maximal in $\mathcal{L}_p({}_RR)$) implies p_0t right irreducible. Moreover, $p_0t|_rbp_0t = a$. But p_0t is not left invertible modulo $\mathrm{r.ann}_R(a)$, otherwise p_0t would be left invertible modulo $J(R)$ (notice that $a \neq 0$, so that $\mathrm{r.ann}_R(a) \subseteq J(R)$), hence p_0t would be invertible, which is a contradiction. The contradiction shows that a is not right hollow. □

Remark 2.16. The factorizations of Proposition 2.6 correspond, in the case of a right noetherian right chain ring, to writing any ordinal number λ in the form $\lambda = \beta + n$, where β is an ordinal with no immediate predecessor, i. e., $\beta = 0$ or β a limit ordinal, and n is a finite ordinal. More precisely, the factorizations of Proposition 2.6 can be chosen to be of the form $a = bp_0^nu = bp_0^{n-1} \cdot pu$ as a product of a right hollow element $b \in R$ and $n \geq 0$ right irreducible elements $p_0, p_0, \dots p_0, p_0u$ with u invertible. Here R is a right noetherian right chain ring and $p_0R = J(R)$.

In fact, we have that $aR = I_\lambda$ for some ordinal λ and $\lambda = \beta + n$ for some β with no immediate predecessor. Let $b \in R$ be such that $bR = I_\beta$. Then $I_\lambda = I_\beta J(R)^n$, so that $aR = bRJ(R)^n = bJ(R)^n = bp_0^nR$. This proves that $a = bp_0^nr$ for some $r \in R$. Now, if $r \in J(R)$, then $aR = I_\lambda \subseteq I_\beta J(R)^{n+1}$, which is a contradiction, because we would get that $\alpha \geq \beta + n + 1$. Thus r is invertible in R.

Let us analyze Theorem 2.1 in light of this example. Let a be an element of a right noetherian right chain ring R and

$$\mathcal{F}(a) := \{ (a_1, a_2, \dots, a_n) \mid n \geq 1,\ a_i \in R,\ a_1 a_2 \dots a_n = a\}$$

be the set of all factorizations of a. Then $aR = I_\lambda$ for some ordinal λ. The set

$$\mathcal{C}_a := \{ (aR, I_2, \dots, I_{n-1}, R) \mid n \geq 1, I_j \text{ a principal right ideal of } R\}$$

of all finite chains of principal right ideals from aR to R_R corresponds to the set

$$\mathcal{C}'_a := \{ (\lambda, \lambda_1, \lambda_2, \dots, \lambda_{n-1}, 0) \mid n \geq 1, \lambda \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq 0\}$$

of all finite descending sequences of ordinals $\leq \lambda$. Thus two factorizations of a are equivalent if and only if they correspond to the same finite descending sequence of ordinals.

Notice that a right chain ring R is a local Bézout ring for which $\mathcal{L}_p(R_R)$ is a sublattice of $\mathcal{L}(R_R)$. In particular, $\mathcal{L}_p(R_R)$ is a modular lattice. A ring R is right chain if and only if 0 is a right rigid element of R, if and only if every element of R is right rigid.

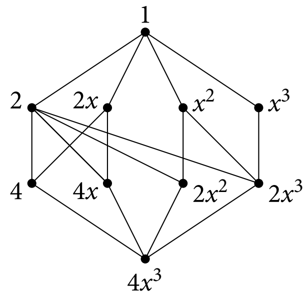
In a right noetherian right chain ring R, the elements not in $p_1 R = \bigcap_{n\geq 0} J(R)^n$ are exactly the elements of R that have a maximal left factorization, unique up to equivalence of factorizations.

Case (b): $[aR, R]_{\mathcal{L}_p(R_R)}$ *is a graded partially ordered set.* Recall that a bounded partially ordered set is graded if and only if all maximal chains have the same finite length. The partially ordered set $[aR, R]_{\mathcal{L}_p(R_R)}$ is graded if and only if all maximal left factorizations of a have the same right length. For instance, consider the subring $R := k[x^2, x^3]$ of the polynomial ring $k[x]$, where k is a field. Every factorization of an element of R in R is a factorization in $k[x]$, which is a commutative PID. It follows that the divisors of $x^6 \in R$ in R are only the polynomials, $\lambda, \lambda x^2, \lambda x^3, \lambda x^4, \lambda x^6$, with $\lambda \in k$, so that the maximal left factorizations of x^6 up to units are only $x^6 = x^2 \cdot x^2 \cdot x^2$ and $x^6 = x^3 \cdot x^3$. Thus $[aR, R]_{\mathcal{L}_p(R_R)}$ is in this case the nonmodular lattice N_5, which is not a graded partially ordered set. For a different example, consider any non-zero element a of a commutative UFD R. If a is the product of t prime elements, then all the maximal chains of $[aR, R]_{\mathcal{L}_p(R_R)}$ have length t, and $[aR, R]_{\mathcal{L}_p(R_R)}$ is a graded partially ordered set.

Case (c): $[aR, R]_{\mathcal{L}_p(R_R)}$ *is a lattice.* This is equivalent to saying that any two left divisors of a always have a left highest common factor and a left least common multiple. It occurs for all commutative UFDs, but also for the commutative ring $R := k[x^2, x^3]$ we have seen in Case (b). This ring R is not a UFD, but for $a = x^6$ the partially ordered set $[aR, R]_{\mathcal{L}_p(R_R)}$ is the lattice N_5.

As far as this Case (c) is concerned, we must recall that, for a commutative integral domain R, $\mathcal{L}_p(R_R)$ is a lattice if and only if every pair of elements of R has a least common multiple, if and only if every pair of elements of R has a greatest common divisor. More generally, if a pair of elements of a commutative integral domain R has a least common multiple, then it has a greatest common divisor. But there exists a commutative integral domain R with a pair of elements that have a greatest common divisor, but not a least common multiple [5, Theorem 2.1].

Here is an example of a ring R with an element a for which $[aR, R]_{\mathcal{L}_p(R_R)}$ is a graded partially ordered set, but not a lattice. Notice that, in general, the partially ordered set $[aR, R]_{\mathcal{L}_p(R_R)}$ has length 0 if an only if a is right invertible. If the partially ordered set $[aR, R]_{\mathcal{L}_p(R_R)}$ has length ≤ 2, then $[aR, R]_{\mathcal{L}_p(R_R)}$ is trivially a lattice. Therefore for an example of a ring R with an element a for which $[aR, R]_{\mathcal{L}_p(R_R)}$ is a graded partially ordered set, but not a lattice, the partially ordered set $[aR, R]_{\mathcal{L}_p(R_R)}$ must have length ≥ 3. Let $\mathbb{Z}[x]$ be the ring of polynomials with integer coefficients and $R := \mathbb{Z}[2x, x^2, x^3]$ be its subring of all polynomials with even coefficient of x. In this ring R, the polynomial $4x^3$ has 10 divisors, up to associates: $1, 2, 4, 2x, 4x, x^2, 2x^2, x^3, 2x^3, 4x^3$. We leave to the reader to check that the partially ordered set $[4x^3 R, R]_{\mathcal{L}_p(R_R)}$ has the following form:



Hence $[4x^3R, R]_{\mathcal{L}_p(R_R)}$ is a graded partially ordered set of length 3, but is not a lattice.

Case (d): $[aR, R]_{\mathcal{L}_p(R_R)}$ *is a modular lattice.* When the partially ordered set $\mathcal{L}_p(R_R)$ is a modular lattice, then the Schreier–Zassenhaus Theorem can be applied, so that any two finite chains between aR and R in $\mathcal{L}_p(R_R)$ have equivalent refinements. Notice that, here, "equivalent refinements" does not mean equivalent in the sense we have used in Theorem 2.1, but it means equivalent refinements in the terminology of modular lattices [20, Section III.3]. Let us be more precise on this point. Let R be an arbitrary ring, $a \in R$ an element and suppose that the interval $[aR, R]_{\mathcal{L}_p(R_R)}$ of all principal right ideals of R between aR and R is a modular lattice. We will denote by $\vee$ and $\wedge$ the least upper bound and the greatest lower bound in this modular lattice. If $aR \subseteq bR$ and $aR \subseteq cR$, then the generators of the principal right ideal $bR \vee cR$ are exactly the left highest common factors of b and c, and the generators of the principal right ideal $bR \wedge cR$ are exactly the left least common multiples of b and c. In this case, it is possible to prove that:

Proposition 2.17. *Let R be a ring and $a \in R$ an element such that $[aR, R]_{\mathcal{L}_p(R_R)}$ is a modular lattice. Let $(c_1, c_2 \dots, c_m)$ and $(d_1, d_2, \dots, d_n)$ be two elements of $\mathcal{F}(a)$. Then there exists $(X_1, \dots, X_n) \in \mathcal{F}(a)$ equivalent to $(d_1, d_2, \dots, d_n)$ and $(Y_1, \dots, Y_m) \in \mathcal{F}(a)$ equivalent to $(c_1, c_2 \dots, c_m)$ with the following property. For $i = 1, \dots, m$ and $j = 1, \dots, n$, there exist elements $x_{i,j}, y_{j,i} \in R$ such that:*
(1) *$X_l = x_{m,n-l+1}x_{m-1,n-l+1} \dots x_{1,n-l+1}$ for all $l = 1, \dots, n$.*
(2) *$Y_k = y_{n,m-k+1}y_{n-1,m-k+1} \dots y_{1,m-k+1}$ for all $k = 1, \dots, m$.*
(3) *For all $i = 1, \dots, m$ and $j = 1, \dots, n$, the intervals*

$$[x_{m,n}x_{m-1,n} \dots x_{1,n}x_{m,n-1} \dots x_{i,j}R, x_{m,n}x_{m-1,n} \dots x_{1,n}x_{m,n-1} \dots x_{i+1,j}R]_{\mathcal{L}_p(R_R)}$$

and

$$[y_{n,m}y_{n-1,m} \dots y_{1,m}y_{n,m-1} \dots y_{j,i}R, y_{n,m}y_{n-1,m} \dots y_{1,m}y_{n,m-1} \dots y_{j+1,i}R]_{\mathcal{L}_p(R_R)}$$

are projective (where $x_{m+1,j} = x_{1,j}$ for $j = 1, \dots, n$ and $y_{n+1,i} = y_{1,i}$ for $i = 1, \dots, m$).

The proof is very similar to the proof of Theorem 2.20 and is therefore omitted. As an example, consider the polynomial ring $R := k[x, y]$ in two commuting indeterminates x, y with coefficients in a field k. This ring R is a commutative UFD that is not a PID. Since R is a UFD, the partially ordered set $[xyR, R]_{\mathcal{L}_p(R_R)}$ is a modular lattice (it is a direct product of finitely many chains of finite length). Consider the two factorizations $xy = yx$. Correspondingly, we have the two chains $xyR \subset xR \subset R$ and $xyR \subset yR \subset R$, in which the intervals $[xyR, xR]_{\mathcal{L}_p(R_R)}$ and $[yR, R]_{\mathcal{L}_p(R_R)}$ are projective and the intervals $[xyR, yR]_{\mathcal{L}_p(R_R)}$ and $[xR, R]_{\mathcal{L}_p(R_R)}$ are projective. All the four intervals, $[xyR, xR]_{\mathcal{L}_p(R_R)}$, $[yR, R]_{\mathcal{L}_p(R_R)}$, $[xyR, yR]_{\mathcal{L}_p(R_R)}$ and $[xR, R]_{\mathcal{L}_p(R_R)}$ are the linearly ordered set with two elements.

For another example, consider an arbitrary ring R. Let S be the multiplicatively closed subset of the center $Z(R)$ of R generated by all prime elements of the commutative ring $Z(R)$, that is, S consists of all elements that are finite products of prime elements of $Z(R)$. Then $[aR, R]_{\mathcal{L}_p(R_R)}$ is a modular lattice of finite length for every $a \in S$ (again, because it is a direct product of finitely many chains of finite length).

Case (e): $[aR, R]_{\mathcal{L}_p(R_R)}$ *is a semimodular lattice of finite length.* Recall that a lattice L is *semimodular* if $x \wedge y <: x$ implies $y <: x \vee y$ for every $x, y \in L$. Here $x <: y$ indicates that y covers x, i. e., $x < y$ and there is no element z such that $x < z < y$. Every modular lattice is semimodular, and every semimodular lattice is a graded partially ordered set [3, p. 40].

In a semimodular lattice, the following result, which is improved in [13], holds:

Theorem 2.18. *If $0 = c_0 <: c_1 <: \cdots <: c_n$ and $0 = d_0 <: d_1 <: \cdots <: d_n$ are two maximal chains in a semimodular lattice with 0 and 1, then there is a permutation π of the set $\{1, 2, \ldots, n\}$ such that $[c_{i-1}, c_i]$ is projective to $[d_{\pi(i)-1}, d_{\pi(i)}]$ for all i.*

As a corollary, we get that:

Corollary 2.19. *Let R be a ring and $a \in R$ an element such that $[aR, R]_{\mathcal{L}_p(R_R)}$ is a semimodular lattice. If $a = a_1 \ldots a_n = b_1 \ldots b_n$ are two maximal left factorizations of a, then there is a permutation π of the set $\{1, 2, \ldots, n\}$ such that $[a_1 \ldots a_{n-i+1}R, a_1 \ldots a_{n-i}R]_{\mathcal{L}_p(R_R)}$ is projective to $[b_1 \ldots b_{n-\pi(i)+1}R, b_1 \ldots b_{n-\pi(i)}R]_{\mathcal{L}_p(R_R)}$ for all i.*

Case (f): $[aR, R]_{\mathcal{L}_p(R_R)}$ *is a sublattice of* $\mathcal{L}(R_R)$. Equivalently, if $b, c \in R$, $aR \subseteq bR$ and $aR \subseteq cR$, then $bR + cR$ and $bR \cap cR$ are principal right ideals. This occurs, for instance, if R is right Bézout right semihereditary. In fact, recall that a ring is *right Bézout* if every finitely generated right ideal is principal. If R is right Bézout right semihereditary, then $bR + cR$ is principal and projective, so that the short exact sequence $0 \to bR \cap cR \to bR \oplus cR \to bR + cR \to 0$ splits. Thus $bR \oplus cR \cong (bR + cR) \oplus (bR \cap cR)$, so that $bR \cap cR$ is finitely generated, hence principal.

Notice that if R is right Bézout right semihereditary, then the matrix ring $\mathbf{M}_n(R)$ is also right Bézout right semihereditary [7, Exercise 0.1.3]. Thus all results apply not only to factorizations of elements of R but also to factorizations of square matrices with entries in R.

Every von Neumann regular ring is right (left) Bézout right (left) semihereditary [11, Theorem 1.1]. In particular, the endomorphism ring of a vector space over a division ring is right (left) Bézout right (left) semihereditary.

If R is a commutative ring, then the polynomial ring $R[x]$ is semihereditary if and only if $R[x]$ is a Bézout ring, if and only if R is von Neumann regular. For a study of the relations between these notions, see [15].

Theorem 2.20. *Let R be a ring and $a \in R$ be an element such that $[aR, R]_{\mathcal{L}_p(R_R)}$ is a sublattice of $\mathcal{L}(R_R)$. Let $(c_1, c_2 \ldots, c_m)$ and $(d_1, d_2, \ldots, d_n)$ be two elements of $\mathcal{F}(a)$. Then, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, there exist elements $x_{i,j}, y_{j,i} \in R$ such that:*

(1) $c_l = x_{m,n-l+1}x_{m-1,n-l+1} \ldots x_{1,n-l+1}$ *for every* $l = 1, \ldots, n$.
(2) $d_k = y_{n,m-k+1}y_{n-1,m-k+1} \cdots y_{1,m-k+1}$ *for every* $k = 1, \ldots, m$.
(3) *For every $i = 1, \ldots, m$ and $j = 1, \ldots, n$, there exists a right R-module isomorphism*

$$\frac{x_{m,n} \ldots x_{1,n}x_{m,n-1} \ldots x_{1,n-1} \ldots x_{i+1,j}R}{x_{m,n} \ldots x_{1,n}x_{m,n-1} \ldots x_{1,n-1} \ldots x_{i+1,j}x_{i,j}R} \cong \frac{y_{n,m} \cdots y_{1,m}y_{n,m-1} \cdots y_{1,m-1} \cdots y_{j+1,i}R}{y_{n,m} \cdots y_{1,m}y_{n,m-1} \cdots y_{1,m-1} \cdots y_{j+1,i}y_{j,i}R}, \tag{1}$$

where $x_{m+1,j} = x_{1,j}$ for $j = 1, \ldots, n$ and $y_{n+1,i} = y_{1,i}$ for $i = 1, \ldots, m$.

Proof. Consider the two chains of principal ideals associated to the given factorizations of a:

$$aR = c_1 \ldots c_m R \subseteq c_1 \ldots c_{m-1}R \subseteq \cdots \subseteq c_1 R \subseteq c_0 R = R_R$$

and

$$aR = d_1 \ldots d_n R \subseteq d_1 \ldots d_{n-1}R \subseteq \cdots \subseteq d_1 R \subseteq d_0 R = R_R,$$

where we define $c_0 = d_0 = 1$. Since $[aR, R]_{\mathcal{L}_p(R_R)}$ is a sublattice of $\mathcal{L}(R_R)$, both the sum and the intersection of any two principal right ideals containing aR are principal. Now, for all $i = 1, \dots, m$ and $j = 1, \dots, n$, let $a_{i,j}$ be a generator of the principal ideal

$$(c_1 \dots c_{m-i}R \cap d_1 \dots d_{n-j}R) + d_1 \dots d_{n-j+1}R$$

and $b_{j,i}$ a generator of the principal ideal

$$(d_1 \dots d_{n-j}R \cap c_1 \dots c_{m-i}R) + c_1 \dots c_{m-i+1}R.$$

We then have [20, Proposition III.3.1] the chain of principal ideals

$$aR \subseteq a_{1,1}R \subseteq \dots \subseteq a_{m,1}R \subseteq a_{1,2}R \subseteq \dots \subseteq a_{m,2}R \subseteq \dots \subseteq a_{m,n}R = R_R \quad (2)$$

with $a_{m,j}R = d_1 \dots d_{n-j}R$ for every $j = 1, \dots, n$, and the chain

$$aR \subseteq b_{1,1}R \subseteq \dots \subseteq b_{n,1}R \subseteq b_{1,2}R \subseteq \dots \subseteq b_{n,2} \subseteq \dots \subseteq b_{n,m}R = R_R$$

with $b_{n,i}R = c_1 \dots c_{m-i}R$ for every $i = 1, \dots, m$. Here we define $a_{0,j} = a_{m,j-1}$ for $j = 1, \dots, n$ and $b_{0,i} = b_{n,i-1}$ for $i = 1, \dots, m$, and we set $b_{0,1} = a_{0,1} = a$. Note that, without loss of generality, we can assume $a_{m,n} = b_{n,m} = 1$, $a_{m,j} = d_1 \dots d_{n-j}$ for every $j = 1, \dots, n$ and $b_{n,i} = c_1 \dots c_{m-i}$ for every $i = 1, \dots, m$. Now, for every $l = 1, \dots, m$ and $j = 1, \dots, n$, there exists an element $x_{l,j}$ such that

$$a_{l,j}x_{l,j} = a_{l-1,j}. \quad (3)$$

Similarly, for every $k = 1, \dots, n$ and $i = 1, \dots, m$, there exists an element $y_{k,i}$ such that

$$b_{k,i}y_{k,i} = b_{k-1,i}. \quad (4)$$

In particular, we have that, for all $j = 1, \dots, n$,

$$a_{m,j+1}x_{m,j+1} = d_1 \dots d_{n-j-1}x_{m,j+1} = a_{m-1,j+1}.$$

Inductively, we get

$$d_1 \dots d_{n-j-1}x_{m,j+1}x_{m-1,j+1} \dots x_{1,j+1} = d_1 \dots d_{n-j}.$$

We then obtain

$$a = d_1 \dots d_n = d_1 \dots d_{n-1}x_{m,1} \dots x_{1,1} = d_1 \dots d_{n-2}x_{m,2} \dots x_{1,2}x_{m,1} \dots x_{1,1},$$

from which we get the factorization

$$a = x_{m,n} \dots x_{1,n}x_{m,n-1} \dots x_{1,n-1}x_{m,n-2} \dots x_{m,1} \dots x_{1,1}. \quad (5)$$

With the same argument, we obtain another factorization of a into mn elements given by

$$a = y_{n,m} \dots y_{1,m}y_{n,m-1} \dots y_{1,m-1}y_{n,m-2} \dots y_{n,1} \dots y_{1,1}.$$

Note that $a_{m,n} = 1$ and (3) imply that $a_{m-1,n} = x_{m,n}$. By applying (3) once again, we have $a_{m-2,n} = x_{m,n}x_{m-1,n}$. Inductively, we obtain that, for all $i = 0, \dots, m$ and $j = 1, \dots, n$,

$$a_{i,j} = x_{m,n}x_{m-1,n} \dots x_{1,n}x_{m-1,n} \dots x_{i+1,j}, \quad (6)$$

where $x_{m+1,j} = x_{1,j}$ for $j = 1, \dots, n$. Similarly, starting from $b_{n,m} = 1$ and applying (4) repeatedly, we get that, for all $i = 1, \dots, m$ and $j = 0, \dots, n$,

$$b_{j,i} = y_{n,m}x_{n-1,m} \dots x_{1,m}x_{n,m-1} \dots x_{j+1,i}, \quad (7)$$

where $y_{n+1,i} = y_{1,i}$ for $i = 1, \dots, m$.

By [20, Proposition III.3.1], the intervals $[a_{i-1,j}R, a_{i,j}R]_{\mathcal{L}_p(R_R)}$ and $[b_{j-1,i}R, b_{j,i}R]_{\mathcal{L}_p(R_R)}$ are projective for all $i = 1, \dots, m$ and $j = 1, \dots, n$. By the observation in [20, Example III.2.1], this implies that the modules $a_{i,j}R/a_{i-1,j}R$ and $b_{j,i}R/b_{j-1,i}R$ are isomorphic for all $i = 1, \dots, m$ and $j = 1, \dots, n$. Combining this with equations (6) and (7), we finally get (1). □

Example 2.21. Let $R = \mathbb{Z}$ and consider the following two factorizations of the element $0 \in \mathbb{Z}$: $0 = 2 \cdot 0 = (-1) \cdot 3 \cdot 0 \cdot 5$. It is easy to verify that, under the bijection established in Theorem 2.1, these two factorizations correspond, respectively, to the chains of principal right ideals

$$0 \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} \tag{8}$$

and

$$0 \subseteq 0 \subseteq 3\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}. \tag{9}$$

Now consider the following refinements of (8) and (9), respectively:

$$0 \subseteq 0 \subseteq 6\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}, \tag{10}$$

$$0 \subseteq 0 \subseteq 6\mathbb{Z} \subseteq 3\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}. \tag{11}$$

Note that (10) and (11) have the same factors $0, 0, 6\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ up to permutation. There exist refinements of the initial two factorizations corresponding to (10) and (11), respectively. For instance, $0 = 2 \cdot 0 = (1 \cdot 2) \cdot (3 \cdot 0 \cdot 0)$ and $0 = (-1) \cdot 3 \cdot 0 \cdot 5 = (-1) \cdot 3 \cdot (2 \cdot 0) \cdot 5$.

Recall that two elements a, b of an arbitrary ring R are said to be *right similar* if the cyclically presented right R-modules $R/aR, R/bR$ are isomorphic, *left similar* if the cyclically presented left R-modules $R/Ra, R/Rb$ are isomorphic [10].

If the element a considered in Theorem 2.20 is right regular and all left divisors of a are also right regular (for example if R is right saturated), we can apply cancellation to the isomorphisms in (1) to obtain the following result: given two factorizations of a in R there exist refinements of the first and the second factorization, respectively, whose factors are the same up to order of factors and right similarity. More precisely, we have the following.

Corollary 2.22. *Let R be a right saturated ring and $a \in R$ be a right regular element such that $[aR, R]_{\mathcal{L}_p(R_R)}$ is a sublattice of $\mathcal{L}(R_R)$. Let $(c_1, c_2 \dots, c_m)$ and $(d_1, d_2, \dots, d_n)$ be two elements of $\mathcal{F}(a)$. Then, for $i = 1, \dots, m$ and $j = 1, \dots, n$, there exist elements $x_{i,j}, y_{j,i} \in R$ such that:*
(1) *$c_l = x_{m,n-l+1}x_{m-1,n-l+1} \dots x_{1,n-l+1}$ for all $l = 1, \dots, n$*
(2) *$d_k = y_{n,m-k+1}y_{n-1,m-k+1} \dots y_{1,m-k+1}$ for all $k = 1, \dots, m$*
(3) *x_{ij} is right similar to y_{ji} for all $i = 1, \dots, m$ and $j = 1, \dots, n$.*

An application of this result will be given below, in Case (h), to the ring $\mathbf{M}_n(\mathbb{Z})$ of $n \times n$ matrices with integer coefficients.

Recall the following easy fact.

Lemma 2.23. *A right Bézout domain R is right semihereditary.*

Proof. Let I be a finitely generated right ideal of R. Then I is a principal right ideal, hence isomorphic either to R_R or to 0_R. Therefore I is a projective right R-module. □

Proposition 2.24. *Let R be a right Bézout domain and $a \in R$ a non-zero element. Let $(c_1, c_2 \dots, c_m)$ and $(d_1, d_2, \dots, d_n)$ be two elements of $\mathcal{F}(a)$. Then, for $i = 1, \dots, m$ and $j = 1, \dots, n$, there exist elements $x_{i,j}, y_{j,i} \in R$ such that:*
(1) *$c_l = x_{m,n-l+1}x_{m-1,n-l+1} \dots x_{1,n-l+1}$ for all $l = 1, \dots, n$*
(2) *$d_k = y_{n,m-k+1}y_{n-1,m-k+1} \dots y_{1,m-k+1}$ for all $k = 1, \dots, m$*

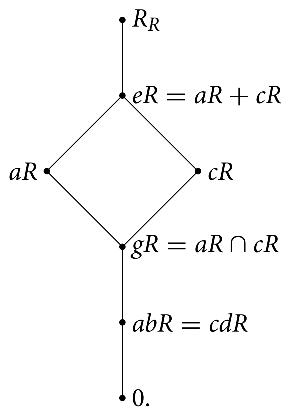
(3) *x_{ij} is right similar and left similar to y_{ji} for all $i = 1, \dots, m$ and $j = 1, \dots, n$.*

Proof. By Lemma 2.23 we have that every non-zero element of a right Bézout domain R satisfies the hypothesis of Corollary 2.22. The result follows after recalling the well-known fact that two elements of an integral domain R are right similar if and only if they are left similar [4, Corollary 1]. □

Proposition 2.25. *Let R be a right Bézout domain, and consider two factorizations $ab = cd \neq 0$ in R. Then a, b, c, d can be factorized in R as $a = ea', b = b'f, c = ec', d = d'f$, with a' right similar and left similar to d' and b' right similar and left similar to c'.*

Proof. Let a, b, c, d be elements of a right Bézout domain R with $ab = cd \neq 0$. Then the sum $aR + cR$ is a principal right ideal of R, $aR + cR = eR$, say. Also, $aR \cap cR$ is a principal right ideal of R (first paragraph of this Case (f)), so $aR \cap bR = gR$, say. Moreover, $aR \cap cR = gR \supseteq abR$. From the inclusions $abR \subseteq gR \subseteq aR \subseteq eR$, we get that $ab = gf$, $g = ab'$ and $a = ea'$ for a suitable $f, b', a' \in R$. Similarly, from the inclusions $gR \subseteq cR \subseteq eR$, we obtain that $g = cd'$ and $c = ec'$ for suitable non-zero elements $d', c' \in R$. We then have that $ab = gf = ab'f$, so that $b = b'f$. Similarly, $cd = ab = gf = cd'f$, from which $d = d'f$.

We have the diagram



Then $eR/aR = eR/ea'R \cong R/a'R, cR/fR = cR/cd'R \cong R/d'R$ and $eR/aR = aR{+}cR/aR \cong cR/aR{\cap}cR = cR/gR$. It follows that $R/a'R \cong R/d'R$. Similarly, by symmetry, $R/c'R \cong R/d'R$. □

In the statement of Proposition 2.25, we thus have that any factorization $ab = cd$ can be refined to a factorization $(ea')(b'f) = (ec')(d'f)$, with a' right similar and left similar to d' and d' right similar and left similar to c'.

Case (g): $[aR, R]_{\mathcal{L}_p(R_R)}$ *is a sublattice of finite length of* $\mathcal{L}(R_R)$.

Corollary 2.26. *Let a be a right regular element of a right Bézout, right semihereditary, right saturated ring R. If $a = a_1a_2 \dots a_n = b_1b_2 \dots b_m$ are any two maximal left factorizations of a (equivalently, two factorizations of a with $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m$ left irreducible elements of R), then $n = m$ and there exists a permutation σ of $\{1, 2, \dots, n\}$ such that a_i is right similar to $b_{\sigma(i)}$ for every $i = 1, 2, \dots, n$.*

Proof. The maximal left factorizations are exactly the factorizations of a that consist of left irreducible elements by Proposition 2.5. The result is an immediate consequence of Corollary 2.22. □

Corollary 2.27. *Let a be a non-zero non-invertible element of a right Bézout domain R. If $a = a_1a_2 \dots a_n = b_1b_2 \dots b_m$ are any two maximal left factorizations of a (equivalently, two factorizations of a with $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m$ irreducible elements of R), then $n = m$ and there exists a permutation σ of $\{1, 2, \dots, n\}$ such that a_i is right similar and left similar to $b_{\sigma(i)}$ for every $i = 1, 2, \dots, n$.*

Proof. We can write "irreducible" by Lemma 2.7. The rest of the statement follows from Proposition 2.24. □

Notice that a non-zero non-invertible element of a right Bézout domain R does not necessarily have a maximal left factorization. More is true: a non-zero non-invertible element of a right PID does not necessarily have a maximal left factorization, as Example 2.11 of a right noetherian right chain domain shows. Every non-zero element of a (right and left) PID has a maximal left factorization, equivalently a factorization into left irreducible elements (see Corollary 2.10 or [16, Theorem 5]).

Case (h): *$[aR, R]_{\mathcal{L}_p(R_R)}$ is a sublattice of $\mathcal{L}(R_R)$ and is a direct product of finitely many chains.* This is the best case in our hierarchy. Recall that direct product decompositions of finite partially ordered sets are not unique up to isomorphism, that is, the Krull-Schmidt Theorem does not hold for finite partially ordered sets [18], but does hold for finite partially ordered sets with a least element and a greatest element [14]. In particular, the partially ordered set $[aR, R]_{\mathcal{L}_p(R_R)}$, when it decomposes as a direct product of chains, has a unique such decomposition. By "direct product" of a family $\{ L_\lambda \mid \lambda \in \Lambda\}$ of partially ordered sets, we mean the cartesian product $\prod_{\lambda\in\Lambda} L_\lambda$ with the component-wise order $(\dots, x_\lambda, \dots) \leq (\dots, y_\lambda, \dots)$ if and only if $x_\lambda \leq y_\lambda$ for every $\lambda \in \Lambda$.

Consider for instance the case of the ring $\mathbb{Z}$. On one hand, for $a = 0$, the interval $[0, \mathbb{Z}]_{\mathcal{L}_p(\mathbb{Z}_\mathbb{Z})}$ is not a product of chains, because there are not, in $[0, \mathbb{Z}]_{\mathcal{L}_p(\mathbb{Z}_\mathbb{Z})}$, two non-zero elements $n\mathbb{Z}, m\mathbb{Z}$ whose intersection is 0. On the other hand, for $a \neq 0$, we have that $a = \pm p_1^{n_1} \dots p_t^{n_t}$, and the interval $[a\mathbb{Z}, \mathbb{Z}]_{\mathcal{L}_p(\mathbb{Z}_\mathbb{Z})}$ is isomorphic to the direct product $\{0, \dots, n_1\} \times \dots \times \{0, \dots, n_t\}$ of t chains $\{0, \dots, n_i\}$, where $0 < \dots < n_i$.

Now consider another example. Let our ring R be the ring $\mathbf{M}_n(\mathbb{Z})$ of all $n \times n$ matrices with entries in $\mathbb{Z}$. We will denote by $0_{n\times n}$ and $1_{n\times n}$, respectively, the zero and the identity element of $\mathbf{M}_n(\mathbb{Z})$. We write $\operatorname{diag}(a_1, \dots, a_n)$ for the diagonal matrix in $\mathbf{M}_n(\mathbb{Z})$ with diagonal entries $a_1, \dots, a_n$.

Recall that the ring $\mathbf{M}_n(\mathbb{Z})$ is a right principal ideal ring and a left principal ideal ring [19, Theorem II.5].

Lemma 2.28. *A matrix $A \in \mathbf{M}_n(\mathbb{Z})$ is a right regular element of $\mathbf{M}_n(\mathbb{Z})$ if and only if it is a left regular element, if and only if* $\det A \neq 0$.

Proof. Let $A \in \mathbf{M}_n(\mathbb{Z})$ be a right regular element of $\mathbf{M}_n(\mathbb{Z})$. Assume that $\det A = 0$. Then there exist $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$ such that $A \begin{pmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$. Then $A \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ \cdot & 0 & \dots & 0 \\ \cdot & 0 & & 0 \\ \alpha_n & 0 & \dots & 0 \end{pmatrix} = 0_{n\times n}$, and this is a contradiction. For the converse, assume $\det A \neq 0$. Then A is invertible in $\mathbf{M}_n(\mathbb{Q})$, i. e., there exists $B \in \mathbf{M}_n(\mathbb{Q})$ such that $BA = 1_{n\times n}$. Hence there exists $C \in \mathbf{M}_n(\mathbb{Z})$ such that $CA = m1_{n\times n}$ for some $m \in \mathbb{Z}$, $m \neq 0$. Now, if $X \in \mathbf{M}_n(\mathbb{Z})$ is such that $AX = 0_{n\times n}$, then $m1_{n\times n}X = CAX = 0_{n\times n}$, which implies $X = 0_{n\times n}$. Therefore A is right regular. The proof that being left regular is equivalent to having non-zero determinant is analogous. □

Corollary 2.29. *The ring $\mathbf{M}_n(\mathbb{Z})$ is right saturated.*

The proof of the following lemma is trivial.

Lemma 2.30. *A matrix $A \in \mathbf{M}_n(\mathbb{Z})$ is a right invertible element of $\mathbf{M}_n(\mathbb{Z})$ if and only if it is a left invertible element, if and only if $|\det A| = 1$.*

Lemma 2.31. *A matrix $A \in \mathbf{M}_n(\mathbb{Z})$ is a right irreducible element of $\mathbf{M}_n(\mathbb{Z})$ if and only if it is a left irreducible element, if and only if $|\det A|$ is a prime number.*

Proof. If $A \in \mathbf{M}_n(\mathbb{Z})$ is a matrix with prime determinant, then for every factorization $A = BC$ in $\mathbf{M}_n(\mathbb{Z})$, either $|\det B| = 1$ or $|\det C| = 1$, hence, by Lemma 2.30, either B or C is invertible. Therefore A is a left irreducible element of $\mathbf{M}_n(\mathbb{Z})$ by Corollary 2.29 and [10, Lemma 6]. For the converse, let $A \in \mathbf{M}_n(\mathbb{Z})$ be a left irreducible matrix. Recall that there exists a factorization $A = UDV$, where U and V are invertible matrices in $\mathbf{M}_n(\mathbb{Z})$ and D is a diagonal matrix with the elementary divisors of A on the diagonal (Smith normal form, see [19, Theorem II.9]). If the determinant of D is not a prime number, it is easy to factorize D as a product of two diagonal noninvertible matrices, say $D = ST$. Hence, if $R = \mathbf{M}_n(\mathbb{Z})$, we have that AR, USR and R are three distinct elements in $[AR, R]_{\mathcal{L}_p(R_R)}$, which contradicts left irreducibility of A. A similar argument proves that, for a regular element of $\mathbf{M}_n(\mathbb{Z})$, being right irreducible is equivalent to having determinant whose absolute value is a prime number. □

As we have already recalled above, two elements of an integral domain are right similar if and only if they are left similar, so that over an integral domain R the notion of similarity is left-right symmetric [4]. The same turns out to be true for regular elements of an arbitrary ring R, as we argue below. (Recall that an element of a ring R is called *regular* if it is both left regular and right regular).

In what follows, for an element c of a ring R, we write λ_c for the right R-module morphism induced by left multiplication by c, and ρ_c for the left R-module morphism induced by right multiplication by c.

Lemma 2.32. *Let a, b be elements of a ring R.*

(1) *If $f: R/aR \to R/bR$ is a right R-module morphism, then there exist $c, c' \in R$ such that $f = \lambda_c$ and $ca = bc'$. Moreover, $\rho_{c'}: R/Rb \to R/Ra$ is a left R-module morphism.*
(2) *If $g: R/Rb \to R/Ra$ is a left R-module morphism, then there exist $c, c' \in R$ such that $g = \rho_{c'}$ and $ca = bc'$. Moreover, $\lambda_c: R/aR \to R/bR$ is a right R-module morphism.*
(3) *Assume that a and b are right regular and that the right R-module morphism $\lambda_c: R/aR \to R/bR$ (with $ca = bc'$ for some $c' \in R$) has an inverse $\lambda_d: R/bR \to R/aR$ (so that there exists $d' \in R$ with $db = ad'$). Then $\rho_{d'}: R/Ra \to R/Rb$ is the inverse of $\rho_{c'}: R/Rb \to R/Ra$. Moreover, there exist $r, s \in R$ such that $cd - 1 = br$, $dc - 1 = as$, $c'd' - 1 = rb$ and $d'c' - 1 = sa$.*
(4) *Assume that a and b be are left regular and that the left R-module morphism $\rho_{c'}: R/Rb \to R/Ra$ (with $ca = bc'$ for some $c \in R$) has an inverse $\rho_{d'}: R/Ra \to R/Rb$ (so that there exists $d \in R$ with $db = ad'$). Then $\lambda_d: R/bR \to R/aR$ is the inverse of $\lambda_c: R/aR \to R/bR$. Moreover, there exists $r, s \in R$ such that $cd - 1 = br$, $dc - 1 = as$, $c'd' - 1 = rb$ and $d'c' - 1 = sa$.*

Proof.

(1) and (2) are straightforward. We only prove (3), since the proof of (4) is analogous. We know that $\lambda_c \circ \lambda_d = id_{R/bR}$ and $\lambda_d \circ \lambda_c = id_{R/aR}$, so that there exist $r, s \in R$ such that $cd - 1 = br$ and $dc - 1 = as$. From this, we get $cdb - b = brb$, so that $cad' - b = brb$, hence $bc'd' - b = brb$. Since b is right regular, we have $c'd' - 1 = rb$. Similarly, $dc - 1 = as$ implies $dca - a = asa$, so that $dbc' - a = asa$. Thus $ad'c' - a = asa$, from which we get $d'c' - 1 = sa$ by right regularity of a. In particular, we have $c'd' - 1 \in Rb$ and $d'c' - 1 \in Ra$, which proves that $\rho_{d'} \circ \rho_{c'} = id_{R/Rb}$ and $\rho_{c'} \circ \rho_{d'} = id_{R/Ra}$. □

Corollary 2.33. *Let a, b be elements of a ring R.*

(1) *If a, b are right regular, then $R/aR \cong R/bR \Rightarrow R/Ra \cong R/Rb$.*
(2) *If a, b are left regular, then $R/Ra \cong R/Rb \Rightarrow R/aR \cong R/bR$.*
(3) *If a, b are regular, then $R/Ra \cong R/Rb \Leftrightarrow R/aR \cong R/bR$.*

The following definition therefore makes sense. We say that two regular elements a, b of a ring R are *similar* if they are right similar (if and only if they are left similar).

Lemma 2.34. *Let $A, B \in \mathbf{M}_n(\mathbb{Z})$ be irreducible elements. Suppose* $\det A = p$ *and* $\det B = q$, *with $p, q \in \mathbb{Z}$ and $|p|, |q|$ primes. Then A is similar to B if and only if $|p| = |q|$.*

Proof. Using the Smith normal form decomposition, we have that $A = LPM$ and $B = L'QM'$ for $P = \operatorname{diag}(1, 1, \dots, 1, p) \in \mathbf{M}_n(\mathbb{Z})$, $Q = \operatorname{diag}(1, 1, \dots, 1, q) \in \mathbf{M}_n(\mathbb{Z})$ and $L, M, L', M' \in \mathbf{M}_n(\mathbb{Z})$ invertible matrices. Now, the morphism $\lambda_{L^{-1}}$ (left multiplication by L^{-1}) induces a right $\mathbf{M}_n(\mathbb{Z})$-module isomorphism between $\mathbf{M}_n(\mathbb{Z})/A\mathbf{M}_n(\mathbb{Z})$ and $\mathbf{M}_n(\mathbb{Z})/P\mathbf{M}_n(\mathbb{Z})$. Analogously, we have an isomorphism between $\mathbf{M}_n(\mathbb{Z})/B\mathbf{M}_n(\mathbb{Z})$ and $\mathbf{M}_n(\mathbb{Z})/Q\mathbf{M}_n(\mathbb{Z})$. Now, A is similar to B if and only if $\mathbf{M}_n(\mathbb{Z})/A\mathbf{M}_n(\mathbb{Z}) \cong \mathbf{M}_n(\mathbb{Z})/B\mathbf{M}_n(\mathbb{Z})$ if and only if $\mathbf{M}_n(\mathbb{Z})/P\mathbf{M}_n(\mathbb{Z}) \cong \mathbf{M}_n(\mathbb{Z})/Q\mathbf{M}_n(\mathbb{Z})$. Since the first module is annihilated by p but not by q and the second one is annihilated by q and not by p, we have that such an isomorphism exists if and only if $|p| = |q|$. $\square$

Proposition 2.35. *Every matrix $A \in \mathbf{M}_n(\mathbb{Z})$ with non-zero determinant is a product of irreducible matrices. This factorization is unique up to the order of factors and similarity.*

Proof. Using the Smith normal form decomposition, we have that $A = UPV$, where P is a diagonal matrix and U, V are invertible matrices in $\mathbf{M}_n(\mathbb{Z})$. It is easy to factorize the matrix P as a product of matrices $P = P_1 \dots P_m$ where $|\det P_i|$ is prime for $i = 1, \dots, m$. We have that $A = (UP_1)P_2 \dots (P_m V)$ is a factorization of A as a product of irreducible matrices. Now, let $A = A_1 \dots A_m = B_1 \dots B_r$ be two factorizations of A into irreducible elements of $\mathbf{M}_n(\mathbb{Z})$. By Lemma 2.31, all A_i's and B_j's have determinant whose absolute value is prime. Therefore, uniqueness of the prime factorization of $|\det A|$ in $\mathbb{N}$ implies that $m = r$ and that there exists a permutation σ of $\{1, \dots, n\}$ such that $|\det A_i| = |\det B_{\sigma(i)}|$ for all $i = 1, \dots, m$. The result now follows from Lemma 2.34. $\square$

Note that Proposition 2.35 can also be derived from Corollary 2.26.

Example 2.36. Consider the following two factorizations of the element $A = \begin{pmatrix} 3 & 3 \\ -3 & 0 \end{pmatrix}$ as product of irreducible elements of $\mathbf{M}_2(\mathbb{Z})$:

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -3 & 0 \end{pmatrix}.$$

We have that both $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ -3 & 0 \end{pmatrix}$ are similar to $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$. However, neither $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ nor $\begin{pmatrix} 1 & 1 \\ -3 & 0 \end{pmatrix}$ is left associate to $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$, since

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \notin \mathbf{M}_2(\mathbb{Z}) \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \notin \mathbf{M}_2(\mathbb{Z}).$$

Proposition 2.35 gives a uniqueness result for factorizations of regular elements of $\mathbf{M}_n(\mathbb{Z})$. As far as singular matrices are concerned, we can apply Theorem 2.20, which implies that, given two factorizations of an element $A \in \mathbf{M}_n(\mathbb{Z})$, there exist refinements of the two factorizations such that the corresponding chains of principal right ideals of $\mathbf{M}_n(\mathbb{Z})$ (Theorem 2.1) have the same factors up to isomorphism and order of factors.

Example 2.37. Consider the following two factorizations in $R := M_2(\mathbb{Z})$

$$\begin{pmatrix} 2 & -2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

It is readily verified that, under the bijection established in Theorem 2.1, the two factorizations above correspond, respectively, to the two chains of principal right ideals of $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$

$$\begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \tag{12}$$

and

$$\begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}. \tag{13}$$

In this particular case, (12) is already a refinement of (13). Hence there exists a refinement of the second factorization that corresponds to the chain (12). One example of such a refinement is

$$\begin{pmatrix} 2 & -2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If, for a matrix $A \in \mathbf{M}_n(\mathbb{Z})$, $|\det A|$ is a prime number p, then A is an irreducible element of $R := \mathbf{M}_n(\mathbb{Z})$ (Lemma 2.31), so that $[AR, R]_{\mathcal{L}_p(R_R)}$ is a chain with two elements. But for $\det A = p^2$, then $[AR, R]_{\mathcal{L}_p(R_R)}$, which is a sublattice of $\mathcal{L}(R_R)$, is not necessarily a direct product of chains. Consider for instance the matrix $A := \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \in \mathbf{M}_2(\mathbb{Z}) = R$. Then $AR = \mathbf{M}_2(p\mathbb{Z})$. The right ideals of $\mathbf{M}_2(\mathbb{Z})$ that contain $\mathbf{M}_2(p\mathbb{Z})$ are in one-to-one correspondence with the right ideals of the ring $\mathbf{M}_2(\mathbb{Z}/p\mathbb{Z})$, which in turn are in one-to-one correspondence with the vector subspaces of the two-dimensional vector space $(\mathbb{Z}/p\mathbb{Z})^2$ over the field $\mathbb{Z}/p\mathbb{Z}$. Clearly, the two-dimensional vector space $(\mathbb{Z}/p\mathbb{Z})^2$ over $\mathbb{Z}/p\mathbb{Z}$ has one vector subspace of dimension 0, one vector subspace of dimension 2, and $p+1$ vector subspaces of dimension $p+1$. Thus the partially ordered set $[AR, R]_{\mathcal{L}_p(R_R)}$, which is a sublattice of $\mathcal{L}(R_R)$, is the modular lattice M_{p+1}, which is not a direct product of chains.

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ORCID

Alberto Facchini http://orcid.org/0000-0002-0144-0193
Martino Fassina http://orcid.org/0000-0003-2810-7216

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