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# Existence and Interior Regularity Theorems for $\bar{\partial}$ on $Q$-Convex Domains

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**Abstract**
We establish for a $q$-convex domain $\Omega \subset \mathbb{C}^n$ existence results in $L^2_{p,k-1}(\Omega, \text{loc})$ and $C^\infty_{p,k-1}(\Omega)$ for the equation $\bar{\partial}u = f$, where $f$ is a $(p, k)$-form on $\Omega$ of degree $k \geq q$ such that $\bar{\partial}f = 0$.

**Keywords** $Q$-convex domains · $\bar{\partial}$-Neumann problem · $L^2$ estimates · Interior regularity

**Mathematics Subject Classification** Primary 32F10

## Introduction

The investigation of the solutions to the Cauchy-Riemann equations $\bar{\partial}u = f$ on a domain $\Omega \subset \mathbb{C}^n$ is one of the foundational problems in the field of several complex variables. Here the datum $f$ is a $(p, k)$-form in $\Omega$ that satisfies the condition $\bar{\partial} f = 0$. When $\Omega$ is pseudoconvex, the existence of a solution is usually proved by considering spaces of forms with square integrable coefficients and looking at $\bar{\partial}$ as a densely defined unbounded operator between Hilbert spaces. One can then exploit functional-analytic techniques and reduce the problem of finding a solution to $\bar{\partial}$ to the much easier task of proving an estimate. This approach dates back to the pioneering works of Morrey [6], Kohn [4,5] and Hörmander [2].

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If $\Omega$ is smooth and bounded, the most efficient proof of the needed estimate exploits the basic Hörmander–Kohn–Morrey formula. One takes advantage of the fact that the boundary integrals appearing in the formula are positive by the assumption of pseudoconvexity. This method also extends to non-smooth and unbounded domains. In these cases, the conclusion is achieved by first proving the existence of a smooth strictly plurisubharmonic exhaustion function $\psi$ and then applying the basic formula to the approximating domains $\Omega_j = \{z \in \Omega : \psi(z) < j\}$. (For a detailed exposition of these techniques we refer to [8, Section 2.11] or [9, Section 1.9]).

Our work starts from the observation that the positivity of the boundary integrals appearing in the Hörmander–Kohn–Morrey formula holds under assumptions on the geometry of $\Omega$ that are weaker than pseudoconvexity. Positivity holds, for instance, on a $q$-convex domain, in the sense of Definition 1.4 below, provided that one considers $(p, k)$-forms with $k \geq q$. It holds, indeed, for more general $q$-pseudoconvex domains (cf. [9, Definition 1.9.4]). If the boundary of $\Omega$ is smooth, then this observation together with the classical theory implies the existence of a solution in $L^2_{p,k-1}(\Omega, \mathrm{loc})$ and $C^\infty_{p,k-1}(\Omega)$ to $\bar\partial u = f$. Recall that here we are assuming $f$ to be a $(p, k)$-form, with $k \geq q$. The case of $\Omega$ non-smooth is more delicate. We show in this paper that it is possible to build a smooth $q$-subharmonic (cf. Definition 1.1 below) exhaustion function out of an upper-semicontinuous one. This in turn allows us to obtain existence results in $L^2$ as well as in the $C^\infty$ setting for the $\bar\partial$-equation in the case of a non-smooth $q$-convex domain $\Omega$. The case of $\Omega$ non-smooth $q$-pseudoconvex is, as far as the authors know, still open.

The paper is organized as follows. In Sect. 1 we present a proof of our main result assuming the existence, for every $q$-convex domain, of a smooth $q$-subharmonic exhaustion function. We then provide an example of a non-smooth $q$-convex domain where the result can be applied. In Sect. 2, after establishing some preliminary results about $q$-subharmonic functions, we show how it is possible to obtain a smooth $q$-subharmonic exhaustion function for a $q$-convex domain.

## 1 The $\bar\partial$-Equation on *Q*-Convex Domains

**Definition 1.1** Let $\Omega$ be an open set in $\mathbb{C}^n$, and let $1 \leq q \leq n$. An upper semicontinuous function $\psi \colon \Omega \to [-\infty, \infty)$ is called *q-subharmonic* if for every $q$-dimensional complex space $W$ in $\mathbb{C}^n$, the restriction $\psi_{|W}$ is a subharmonic function on $W \cap \Omega$. This means that for every compact subset $K \subset\subset W \cap \Omega$ and every continuous harmonic function $h$ on $K$ such that $\psi \leq h$ on $\partial K$, then $\psi \leq h$ on $K$.

***Remark 1.2*** Note that 1-subharmonic functions are plurisubharmonic and $n$-subharmonic functions are subharmonic.

***Remark 1.3*** If $\psi \in C^2(\Omega)$ and $\lambda_1^\psi \leq \cdots \leq \lambda_n^\psi$ are the eigenvalues of the complex Hessian $\partial\bar\partial\psi$, then the $q$-subharmonicity of $\psi$ is equivalent to the condition

$$\sum_{j=1}^{q} \lambda_j^\psi \geq 0. \tag{1.1}$$

If the inequality (1.1) is strict, we say that $\psi$ is *strictly $q$-subharmonic*.

**Definition 1.4** A domain $\Omega \subset \mathbb{C}^n$ is said to be *$q$-convex* if it has a $q$-subharmonic exhaustion function, that is, if there exists a $q$-subharmonic function $\psi$ on $\Omega$ such that for all $j \in \mathbb{R}$ the set $\Omega_j = \{z \in \Omega : \psi(z) < j\}$ is relatively compact in $\Omega$.

***Remark 1.5*** Let $\Omega \subset \mathbb{C}^n$ be a domain with $C^2$ boundary $b\Omega$, and let $\rho$ be a local defining function for $\Omega$. Let $\lambda_1^{b\Omega} \le \cdots \le \lambda_{n-1}^{b\Omega}$ be the eigenvalues of the Levi form $\partial\bar\partial\rho|_{T^{\mathbb{C}}b\Omega}$. It is readily checked that the $q$-convexity of $\Omega$ is characterized by either $\sum_{j=1}^{q} \lambda_j^{b\Omega} \ge 0$ or $\sum_{j=1}^{q} \lambda_j^{\psi} \ge 0$. Here $\psi := -\log d_{b\Omega}(z) + c|z|^2$ for a suitable $c \in \mathbb{R}$, with $d_{b\Omega}(z)$ being the distance function from the boundary.

For a domain $\Omega$ in $\mathbb{C}^n$, we denote by $L^2_{p,k}(\Omega, \text{loc})$ (resp $L^2_{p,k}(\Omega, \text{loc})$) the space of $(p,k)$-forms whose coefficients are square integrable (resp. locally square integrable) in $\Omega$. If $\varphi$ is a continuous function on $\Omega$, then $L^2_{p,k}(\Omega, \varphi)$ denotes the space $(p,k)$-forms on $\Omega$ whose coefficients are square integrable with respect to the weight $e^{-\varphi}$. Moreover, we write $C^\infty_{p,k}(\Omega)$ for the space of $(p,k)$-forms whose coefficients are smooth in $\Omega$. Note that the subscript $(p,k)$ will be dropped whenever there is no danger of confusion. Our main result is the following.

**Theorem 1.6** *Let $\Omega \subset \mathbb{C}^n$ be a $q$-convex domain.*

(1) *For every $f \in L^2_{p,k}(\Omega, loc)$ with $\bar\partial f = 0$ and $k \ge q$, there exists a form $u \in L^2_{p,k-1}(\Omega, loc)$ such that $\bar\partial u = f$.*
(2) *For any $f \in C^\infty_{p,k}(\Omega)$ with $\bar\partial f = 0$ and $k \ge q$, there exists $u \in C^\infty_{p,k-1}(\Omega)$ such that $\bar\partial u = f$.*

For $\Omega$ pseudoconvex in the usual sense, that is, for $q = 1$, the result is classical and can be found for instance in Hörmander's book [3, Theorem 4.2.2 and Corollary 4.2.6]. For $\Omega$ a $q$-convex domain with smooth boundary the result is also known [1, Theorem 3.1 and Corollary 4.2]. In this paper, we remove the smoothness hypothesis on the boundary of $\Omega$.

We now provide an example of a $q$-convex domain $\Omega$ which is not smooth. We can therefore apply Theorem 1.6. The result, however, cannot be recovered from the existing literature.

***Example 1.7*** Consider the space $\mathbb{C}^3$ with coordinates $z_j = x_j + iy_j$ for $j = 1, 2, 3$. Let $\Omega$ be the subset of $\mathbb{C}^3$ defined by

$$\begin{cases} y_3 < -|z_1|^2 + |z_2|^2 \\ y_2 < -|z_1|^2 + |z_3|^2 \\ |z| < \frac{1}{4}. \end{cases}$$

Define the functions

$$f_1 = -\log\big(-y_3 - |z_1|^2 + |z_2|^2\big), \quad f_2 = -\log\big(-y_2 - |z_1|^2 + |z_3|^2\big),$$
$$f_3 = -\log\Big(\frac{1}{16} - |z|^2\Big).$$

A continuous 2-subharmonic exhaustion function for $\Omega$ is given by

$$\psi = \sup\{f_1, f_2, f_3\} + c|z|^2,$$

for a suitable $c \in \mathbb{R}$.

***Remark 1.8*** The more involved problem of the regularity at the boundary for the $\bar{\partial}$-Neumann problem on a $q$-convex domain has been recently solved in [7]: if $\Omega$ is bounded, real analytic and $q$-convex, then the canonical solution $u = \bar{\partial}^* N f$ of $\bar{\partial} u = f$ for $f \in C^\infty_{p,k}(\overline{\Omega})$ belongs to $C^\infty_{p,k-1}(\overline{\Omega})$.

***Proof of Theorem 1.6*** The first step is to obtain a smooth strictly $q$-subharmonic exhaustion function $\psi$ for $\Omega$ starting from an upper semicontinuous one. This is achieved in Theorem 2.6. With such a function $\psi$ available, the statement follows from the proof of [9, Theorem 1.9.14]. For the convenience of the reader, we describe the main steps of the argument for the specific case in which $\psi$ is $q$-subharmonic instead of $q$-plurisubharmonic in the more general sense of [9]. Let $\lambda_1^\psi \leq \cdots \leq \lambda_n^\psi$ be the eigenvalues of the complex Hessian $\partial\bar{\partial}\psi$. Recall that the strict $q$-subharmonicity of $\psi$ is equivalent to the condition $\sum_{j=1}^q \lambda_j^\psi > 0$. Given the equation $\bar{\partial} u = f$, we choose a real convex function $\eta(t)$, $t \in \mathbb{R}$ with $\eta(t) = 0$ when $t \leq 0$ and so rapidly increasing that, with $C(z)$ denoting an upper bound for the $C^2$ norm of $\psi$, the following hold:

$$\begin{cases} f \in L^2(\Omega, \eta(\psi)), \\ \sum_{j=1}^{q} \lambda_j^{\eta(\psi)}(z) \geq 1 + C(z). \end{cases}$$

By [9, Lemma 1.9.6], we have

$$||u||^2_{L^2(\Omega_j,\eta(\psi))} \lesssim ||\bar{\partial} u||^2_{L^2(\Omega_j,\eta(\psi))} + ||\bar{\partial}^*_{\eta(\psi)} u||^2_{L^2(\Omega_j,\eta(\psi))},$$

where $\lesssim$ indicates inequality up to a multiplicative constant and $\bar{\partial}^*_{\eta(\psi)}$ is the adjoint of $\bar{\partial}$ in the weighted space $L^2(\Omega, \eta(\psi))$.
Repeating the argument of [9, Theorem 1.9.7], we obtain a family of solutions $\bar{\partial} u_j = f$ on the approximating domains $\Omega_j$ with the property that

$$||u_j||^2_{L^2(\Omega_j,\eta(\psi))} \lesssim ||f||^2_{L^2(\Omega_j,\eta(\psi))} \leq ||f||^2_{L^2(\Omega,\eta(\psi))}.$$

Taking a weak subsequential limit of the $u_j$, we obtain a solution $u \in L^2(\Omega, \eta(\psi))$ to $\bar{\partial} u = f$. This proves (1). As for (2), by the argument of [9, Theorem 1.9.9], there exists a well defined Neumann operator $N_{\eta(\psi)}$. Setting $u := \bar{\partial}^*_{\eta(\psi)} N_{\eta(\psi)} f$ we have that $\bar{\partial} u = f$ and $\bar{\partial}^*_{\eta(\psi)} u = 0$. By the ellipticity of the system $(\bar{\partial}, \bar{\partial}^*_{\eta(\psi)})$ (cf. [9, Theorem 1.9.8]), we conclude that $u \in C^\infty_{p,k-1}(\Omega)$ if $f \in C^\infty_{p,k}(\Omega)$. □

## 2 *Q*-Subharmonic Functions and *Q*-Convex Domains

We start by recalling a few elementary properties of $q$-subharmonic functions, which we introduced in Definition 1.1.

**Proposition 2.1** *Let $\Omega \subset \mathbb{C}^n$ be an open set, and let $1 \le q \le n$. Then*

(1) *If $\{u_\alpha\}_{\alpha \in A}$ is a family of $q$-subharmonic functions and $u = \sup_{\alpha\in A} u_\alpha < \infty$ is upper semicontinuous, then $u$ is a $q$-subharmonic function in $\Omega$.*
(2) *If $\{u_j\}_{j=1}^{\infty}$ is a decreasing sequence of $q$-subharmonic functions, then the function $u = \lim_{j\to\infty} u_j$ is also $q$-subharmonic.*

Let $S_r^{2q-1}(z)$ and $\mathbb{B}_r^{2q}(z)$ be respectively the sphere and the ball of center $z$ and radius $r$ in a complex $q$-space. We denote by $|S_r^{2q-1}(z)|$ and $|\mathbb{B}_r^{2q}(z)|$ their Euclidean measure. The following result follows immediately from the corresponding statement for subharmonic functions ([9, Theorem 1.4.2]).

**Proposition 2.2** *Let $\Omega \subset \mathbb{C}^n$ be an open set. For an upper semicontinuous function $\psi$ on $\Omega$ the following are equivalent.*

(1) *$\psi$ is $q$-subharmonic in $\Omega$.*
(2) *For any complex $q$-space $W$ in $\mathbb{C}^n$ and every sphere $S_r^{2q-1}(z) \subset\subset W \cap \Omega$*

$$\psi(z) \le \frac{1}{|S_r^{2q-1}(z)|}\int_{S_r^{2q-1}(z)} \psi(\zeta)\, dS(\zeta) \quad \textit{(spherical submean).} \tag{2.1}$$

(3) *For any complex $q$-space $W$ in $\mathbb{C}^n$ and every ball $\mathbb{B}_r^{2q}(z) \subset\subset W \cap \Omega$*

$$\psi(z) \le \frac{1}{|\mathbb{B}_r^{2q}(z)|}\int_{\mathbb{B}_r^{2q-1}(z)} \psi(\zeta)\, dV(\zeta) \quad \textit{(solid submean).} \tag{2.2}$$

Note that the integrals in (2.1) and (2.2) are well defined because of the semicontinuity of $\psi$.

We next recall a useful lemma. Let $\Delta^W$ denote the Laplacian along a complex $q$-space $W$. When $\psi$ is not of class $C^2$, the action of $\Delta^W$ on $\psi$ is meant in the weak sense of distributions.

**Lemma 2.3** *Let $\psi$ be a $q$-subharmonic function on a domain $\Omega \subset \mathbb{C}^n$. Then*

$$\Delta^W \psi \ge 0 \textit{ on } \Omega \cap W \textit{ for every complex } q\textit{-plane } W. \tag{2.3}$$

*Conversely, if $\psi \in C^2(\Omega)$ is such that (2.3) holds, then $\psi$ is $q$-subharmonic in $\Omega$.*

***Proof*** For the first statement, let $W$ be a complex $q$-plane and $v \in C^2(\Omega \cap W)$ a compactly supported function such that $v \ge 0$. We want to prove that

$$\int_{\Omega\cap W} \psi\, \Delta^W v \, d\lambda \ge 0, \tag{2.4}$$

where $\lambda$ denotes the Lebesgue measure. Let $0 < r < \mathrm{dist}(\mathrm{supp}\, v, \mathbb{C}^n \setminus \Omega)$ and $z \in \mathrm{supp}\, v$. By (2.1) we have

$$\psi(z) \leq \frac{1}{|S_1^{2q-1}(0)|} \int_{S_1^{2q-1}(0)} \psi(z + r\zeta)\, dS(\zeta).$$

Multiplying by $v$ and integrating with respect to $z$, we obtain

$$\int_{\Omega \cap W} \psi(z) v(z)\, d\lambda(z) \leq \frac{1}{|S_1^{2q-1}(0)|} \int_{\Omega \cap W} \int_{S_1^{2q-1}(0)} \psi(z + r\zeta)\, dS(\zeta)\, v(z)\, d\lambda(z). \tag{2.5}$$

Performing the change of variable $z' = z + r\zeta$ on the right side of (2.5), we have

$$0 \leq \frac{1}{|S_1^{2q-1}(0)|} \int_{\Omega \cap W} \left( \int_{S_1^{2q-1}(0)} \Big( v(z' - r\zeta) - v(z') \Big) dS(\zeta) \right) \psi(z')\, d\lambda(z'). \tag{2.6}$$

A second order Taylor expansion of $v(\zeta)$ at $z'$ shows that (2.6) is equivalent to

$$0 \leq \int_{\Omega \cap W} \psi(z) \Delta^W v(z) r^2\, d\lambda(z) + o(r^2), \tag{2.7}$$

since the only term surviving after integration is the non-harmonic one. Dividing by $r^2$ and letting $r \to 0$, we see that (2.7) tends to (2.4).

For the converse, assume that (2.3) holds and let $W$ be a complex $q$-plane. Then, denoting as $\psi_W$ the restriction of $\psi$ to $W$, we have

$$\begin{aligned} \frac{\partial}{\partial r} \left( \frac{1}{|S_r^{2q-1}(z)|} \int_{S_r^{2q-1}(z)} \psi_W(\zeta)\, dS(\zeta) \right) &= \frac{\partial}{\partial r} \left( \frac{1}{|S_1^{2q-1}(z)|} \int_{S_1(z)^{2q-1}} \psi(r\zeta)\, dS(\zeta) \right) \\ &= \frac{r}{|S_1^{2q-1}(z)|} \int_{S_1^{2q-1}(z)} \Big( \nabla \psi_W(\zeta) \cdot \zeta \Big)\, dS(\zeta) \\ &= \frac{r}{|S_1^{2q-1}(z)|} \int_{\mathbb{B}_1^{2q}(z)} \Delta^W \psi(r\zeta)\, dV(\zeta) \\ &\geq 0, \end{aligned} \tag{2.8}$$

where we have used the divergence theorem and the hypothesis (2.3). Equation (2.8) implies that the function

$$M(r) := \frac{1}{|S_r^{2q-1}|} \int_{S_r^{2q-1}} \psi_W(\zeta)\, dS(\zeta)$$

is increasing for $r \geq 0$. Note that the inequality $M(0) \leq M(r)$ for $r \geq 0$ corresponds to the submean property (2.1). Hence $\psi$ is $q$-subharmonic by Proposition 2.2. □

We now introduce the notation $\Omega_\epsilon := \{z \in \Omega : d_{b\Omega}(z) > \epsilon\}$ and we consider a smooth radial function $\rho = \rho(|z|)$ with compact support such that $\int_{\mathbb{C}^n} \rho\, dV = 1$.

**Proposition 2.4** *Let $\psi$ be an upper semicontinuous function on a domain $\Omega \subset \mathbb{C}^n$ such that $\Delta^W \psi \geq 0$ on $\Omega \cap W$ for every complex $q$-plane $W$. For $\epsilon > 0$ define*

$$\psi_\epsilon(z) := \int_{\mathbb{C}^n} \psi(z - \epsilon\,\zeta)\,\rho(|\zeta|)\,dV(\zeta), \quad z \in \Omega_\epsilon.$$

*Then $\psi_\epsilon$ is $q$-subharmonic in $\Omega_\epsilon$ for every $\epsilon > 0$. Moreover, $\psi_\epsilon \searrow \psi$. In particular, $\psi$ is $q$-subharmonic in $\Omega$.*

***Proof*** Let $W$ be a complex $q$-plane. It is not restrictive to assume, in coordinates $z = (z', z'')$, that $W$ is defined by $z'' = 0$. Accordingly, we denote $\Delta^W$ by $\Delta_{z'}$. Now,

$$\begin{aligned}\Delta_{z'}\psi_\epsilon &= \Delta_{z'}\left(\int_{W^\perp} dV(\zeta'') \int_W dV(\zeta')\,\psi(\zeta', \zeta'')\,\rho\left(\frac{z' - \zeta'}{\epsilon}, \frac{z'' - \zeta''}{\epsilon}\right)\right) \\ &= \int_{W^\perp} dV(\zeta'') \int_W dV(\zeta')\,\psi(\zeta', \zeta'')\,\Delta_{z'}\rho\left(\frac{z' - \zeta'}{\epsilon}, \frac{z'' - \zeta''}{\epsilon}\right).\end{aligned} \tag{2.9}$$

Since the integrand of (2.9) is non-negative, then $\Delta_{z'}\psi_\epsilon \geq 0$ on $\Omega_\epsilon \cap W$. By Lemma 2.3, $\psi_\epsilon$ is $q$-subharmonic in $\Omega_\epsilon$ for every $\epsilon$. In particular, each $\psi_\epsilon$ is also $n$-subharmonic (that is, subharmonic) and therefore $(\psi_\epsilon)_\delta$ is monotonic. Taking the limit for $\epsilon \to 0$, we obtain that $\psi_\epsilon$ itself is monotonic. By the upper semicontinuity of $\psi$ we have that $\limsup \psi_\epsilon \leq \psi$. Hence $\psi_\epsilon \searrow \psi$. The last statement follows from Proposition 2.1. □

Combining Proposition 2.4 and Lemma 2.3 we immediately obtain the following.

**Corollary 2.5** *For an upper semicontinuous function $\psi$ on a domain $\Omega \subset \mathbb{C}^n$,*

*$\psi$ is $q$-subharmonic in $\Omega \iff \Delta^W \psi \geq 0$ on $\Omega \cap W$ for every complex $q$-plane $W$.*

We have now gathered all the necessary tools to prove the main result of this section. Recall that a domain $\Omega$ is said to be $q$-convex if it has a $q$-subharmonic exhaustion function $\psi$. We now prove that $\psi$ can be taken to be smooth. The proof follows the steps of [3, Theorem 2.6.11].

**Theorem 2.6** *Let $\Omega \subset \mathbb{C}^n$ be a $q$-convex domain. Then $\Omega$ admits a smooth strictly $q$-subharmonic exhaustion function.*

***Proof*** We set $\Omega_j := \{z \in \Omega : \psi(z) < j\}$ and define, for small values of $\epsilon > 0$,

$$\psi_j(z) := \int_{\Omega_{j+1}} \psi(\zeta)\,\rho\left(\frac{z - \zeta}{\epsilon}\right)\epsilon^{-2n}\,dV(\zeta) + \epsilon|z|^2, \quad j = 0, 1, \ldots \tag{2.10}$$

Each function $\psi_j$ is smooth on $\Omega_j$. Moreover, for a suitable choice of $\epsilon$ (depending on $j$) we can apply Proposition 2.4 and obtain

$$\psi_j > \psi \text{ and } \psi_j \text{ strictly } q\text{-subharmonic on } \overline{\Omega}_j. \tag{2.11}$$

We can also achieve (by semicontinuity)

$$\psi_j \le \psi + 1 \text{ on } \overline{\Omega}_j. \tag{2.12}$$

Consider now a real convex function $\chi(t)$, $t \in \mathbb{R}$, with $\chi \equiv 0$ for $t \le 0$ and $\dot{\chi} > 0$ for $t > 0$. Then

$$\begin{cases} \chi(\psi_j + 1 - j) \text{ is stictly } q\text{-subharmonic on } \overline{\Omega}_j \backslash \Omega_{j-1}, \\ \chi(\psi_j + 1 - j) \underset{\text{by (2.12)}}{=} 0 \text{ on } \Omega_{j-2}. \end{cases}$$

We can therefore choose successively positive numbers $a_j \in \mathbb{R}$ so that, if we define

$$\widetilde{\psi}_m := \psi_0 + \sum_{j=1}^{m} a_j \chi(\psi_j + 1 - j),$$

we have $\widetilde{\psi}_m > \psi$ and $\widetilde{\psi}_m$ strictly $q$-subharmonic in $\overline{\Omega}_m$. Since

$$\widetilde{\psi}_m = \widetilde{\psi}_l \text{ on } \Omega_j \text{ for } j < m, l,$$

then the limit $\widetilde{\psi} := \lim_m \widetilde{\psi}_m$ is well defined and satisfies

$$\begin{cases} \widetilde{\psi} > \psi, \\ \widetilde{\psi} \text{ is strictly } q\text{-subharmonic.} \end{cases}$$

□

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