Supnorm estimates for $\overline{\partial}$ on product domains in $\mathbb{C}^n$

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Graduate Analysis Seminar

May 2nd, 2019

Supnorm estimates for $\overline{\partial}$

The problem

Let $\Omega \subset \mathbb{C}^n$ a domain and $f = f_1 d\overline{z}_1 + \cdots + f_n d\overline{z}_n$ a $(0,1)$ form on Ω, $\overline{\partial} f = 0$. Is there a solution to

$$\overline{\partial} u = f \quad \text{in } \Omega$$

satisfying

$$\|u\|_{L^\infty(\Omega)} \leq C(\Omega) \|f\|_{L^\infty(\Omega)} \ ?$$

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The answer depends on

- The geometry of Ω
- The regularity of the f_j

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Some level of smoothness on the datum f is always assumed

Stein's question

What if $\Omega = D_1 \times \cdots \times D_n$ a polydisc in $\mathbb{C}^n$, $f_j \in L^\infty(\Omega)$?

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More precisely

Question (Kerzman, Comm. Pure Appl. Math., 1971)

Let Ω be a polydisc in $\mathbb{C}^n$ and $f = f_1 d\bar{z}_1 + \cdots + f_n d\bar{z}_n$ a $(0,1)$ form on Ω with components $f_j \in L^\infty(\Omega)$. Assume that $\bar{\partial} f = 0$ in weak sense in Ω. Is there $u \in L^\infty(\Omega)$ solving

$$\bar{\partial} u = f \text{ weakly in } \Omega$$

and satisfying the supnorm estimate

$$\|u\|_{L^\infty(\Omega)} \leq C(\Omega) \|f\|_{L^\infty(\Omega)} \text{ ?}$$

The one-dimensional tool

Let $D \subset \mathbb{C}$ be an open bounded domain, $f \in L^1(D)$.

$$Tf(z) := -\frac{1}{2\pi i} \int_D \frac{f(\zeta)}{\zeta - z}\, d\bar{\zeta} \wedge d\zeta.$$

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Theorem 1

$$\partial_{\bar{z}} Tf = f \text{ in weak sense in } D$$

Henkin's solution on the bidisc

Let $\Omega = D_1 \times D_2$ be a polydisc in $\mathbb{C}^2$, $f = f_1 d\bar{z}_1 + f_2 d\bar{z}_2$ a $\bar{\partial}$-closed form on Ω, $f_1, f_2 \in C^1(\overline{\Omega})$.

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Henkin's solution to $\overline{\partial} u = f$ is given by

$$\begin{aligned} u(z_1, z_2) = &- \frac{1}{2\pi i} \int_{D_1} \frac{f_1(\zeta_1, z_2)}{\zeta_1 - z_1} \, d\bar{\zeta}_1 \wedge d\zeta_1 \\ &- \frac{1}{2\pi i} \int_{D_2} \frac{f_2(z_1, \zeta_2)}{\zeta_2 - z_2} \, d\bar{\zeta}_2 \wedge d\zeta_2 \\ &- \frac{1}{(2\pi i)^2} \int_{D_1 \times D_2} \frac{\partial_{\bar{\zeta}_1} f_2(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \, d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_2 \end{aligned}$$

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Recall

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Let $\Omega = D_1 \times \cdots \times D_n$ be a product of 1-dimensional domains. Define the "slice" operator T^k on $f \in L^1(\Omega)$ by

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$$\mathbf{T} f := T^1 f_1 + T^2 f_2 - T^1 T^2 (\partial_{\bar{z}_1} f_2)$$

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$$\begin{aligned} \partial_{\bar{z}_2} \mathbf{T} f &= \partial_{\bar{z}_2} T^1 f_1 + \partial_{\bar{z}_2} T^2 f_2 - \partial_{\bar{z}_2} T^2 T^1 (\partial_{\bar{z}_2} f_1) \\ &= T^1 (\partial_{\bar{z}_2} f_1) + f_2 - T^1 (\partial_{\bar{z}_2} f_1) = f_2 \end{aligned}$$

A solution operator in higher dimension

Let $\Omega = D_1 \times D_2 \times D_3$ be a polydisc in $\mathbb{C}^3$,
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It is clear how to generalize **T** to higher dimension

A solution operator in higher dimension

Proposition 2 (F.-Pan, 2019)

Let $\Omega = D_1 \times \cdots \times D_n$ be the product of open bounded domains with $C^{1,\alpha}$ boundary, where $0 < \alpha < 1$. For a $(0,1)$ form $f = f_1 d\bar{z}_1 + \cdots + f_n d\bar{z}_n$ in Ω with $f_j \in C^{n-1,\alpha}(\Omega)$, we define

$$\mathbf{T}f := \sum_{s=1}^{n} (-1)^{s-1} \sum_{1 \le i_1 < \cdots < i_s \le n} T^{i_1} \ldots T^{i_s} \left(\frac{\partial^{s-1} f_{i_s}}{\partial \bar{z}_{i_1} \ldots \partial \bar{z}_{i_{s-1}}} \right)$$

We have

- $\mathbf{T}f \in C^{1,\alpha}(\Omega)$
- If f is $\bar{\partial}$-closed, then $\bar{\partial}\mathbf{T}f = f$ in Ω

Supnorm estimates for $\mathbf{T}f$

Lemma 3

Let $D \subset \mathbb{C}$ be a an open bounded domain and let $\alpha < 2$. There exists a constant C such that

$$\left\| \int_D \frac{|d\bar{\zeta} \wedge d\zeta|}{|\zeta - z|^\alpha} \right\|_{L^\infty(D)} \leq C$$

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Hence in particular

$$\left\| T^k f \right\|_{L^\infty(\Omega)} \leq C \, \|f\|_{L^\infty(\Omega)}$$

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Enough to argue for the term

$$T^1 T^2(\partial_{\bar{z}_1} f_2) = -\frac{1}{(2\pi i)^2} \int_{D_1 \times D_2} \frac{\partial_{\bar{\zeta}_1} f_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)}$$

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$$\begin{aligned} -(2\pi i)^2 T^1 T^2(\partial_{\bar{z}_1} f_2) &= \int_{D_1\times D_2} \partial_{\bar{\zeta}_2} f_1 \frac{(\overline{\zeta_2 - z_2})}{(\zeta_1 - z_1)|\zeta - z|^2} \\ &\quad + \int_{D_1\times D_2} \partial_{\bar{\zeta}_1} f_2 \frac{(\overline{\zeta_1 - z_1})}{(\zeta_2 - z_2)|\zeta - z|^2} \\ &= I_1 + I_2 \end{aligned}$$

Supnorm estimates for $\mathbf{T}f$: dimension 2

$$I_2 = \int_{D_2} \left\{ \int_{\partial D_1} f_2 \, \frac{(\overline{\zeta_1 - z_1})}{(\zeta_2 - z_2)|\zeta - z|^2} \, d\zeta_1 \right.$$
$$\left. - \int_{D_1} f_2 \, \frac{\partial}{\partial \bar{\zeta}_1} \left[\frac{(\overline{\zeta_1 - z_1})}{(\zeta_2 - z_2)|\zeta - z|^2} \right] d\bar{\zeta}_1 \wedge d\zeta_1 \right\} d\bar{\zeta}_2 \wedge d\zeta_2$$

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We need to estimate

$$\left\| \int_{\partial D_1 \times D_2} \frac{|\overline{\zeta_1 - z_1}|}{|\zeta_2 - z_2||\zeta - z|^2} \right\|_{L^\infty(\Omega)} \qquad \left\| \int_{D_1 \times D_2} \frac{|\overline{\zeta_2 - z_2}|}{|\zeta - z|^4} \right\|_{L^\infty(\Omega)}$$

Supnorm estimates for $\mathbf{T}f$: dimension 2

Lemma 4

Let $D \subset \mathbb{C}$ be a an open bounded domain and let $\alpha < 2$. There exists a constant C such that

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Lemma 5

Let $D \subset \mathbb{C}$ be an open bounded domain with C^1 boundary ∂D and let $\alpha < 1$. There exists a constant C such that

$$\left\| \int_{\partial D} \frac{|d\zeta|}{|\zeta - z|^\alpha} \right\|_{L^\infty(D)} \leq C$$

Supnorm estimates for $\mathbf{T}f$: dimension 2

$$\frac{|\overline{\zeta_1 - z_1}|}{|\zeta_2 - z_2||\zeta - z|^2} \leq \frac{1}{|\zeta_1 - z_1|^{\frac{1}{3}}|\zeta_2 - z_2|^{\frac{5}{3}}}$$

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$$\frac{|\overline{\zeta_2 - z_2}|}{|\zeta - z|^4} \leq \frac{1}{|\zeta_1 - z_1|^{\frac{4}{3}}|\zeta_2 - z_2|^{\frac{5}{3}}}$$

Supnorm estimates for $\mathbf{T}f$: dimension 2

$$\frac{|\overline{\zeta_1 - z_1}|}{|\zeta_2 - z_2||\zeta - z|^2} \leq \frac{1}{|\zeta_1 - z_1|^{\frac{1}{3}}|\zeta_2 - z_2|^{\frac{5}{3}}}$$

$$\frac{|\overline{\zeta_2 - z_2}|}{|\zeta - z|^4} \leq \frac{1}{|\zeta_1 - z_1|^{\frac{4}{3}}|\zeta_2 - z_2|^{\frac{5}{3}}}$$

Hence

$$\left\| \int_{\partial D_1 \times D_2} \frac{|\overline{\zeta_1 - z_1}|}{|\zeta_2 - z_2||\zeta - z|^2} \right\|_{L^\infty(\Omega)} \leq C$$

$$\left\| \int_{D_1 \times D_2} \frac{|\overline{\zeta_2 - z_2}|}{|\zeta - z|^4} \right\|_{L^\infty(\Omega)} \leq C$$

The Main Theorem

Theorem 6 (F.-Pan, 2019)

Let $\Omega = D_1 \times \cdots \times D_n$ be the product of open bounded domains with $C^{1,\alpha}$ boundary, where $0 < \alpha < 1$. Let $f = f_1 d\bar{z}_1 + \cdots + f_n d\bar{z}_n$ be a $\bar\partial$-closed $(0,1)$ form on Ω with components $f_j \in C^{n-1,\alpha}(\Omega)$. There exists $\mathbf{T}f \in C^{1,\alpha}(\Omega)$ solving

$$\bar\partial \mathbf{T} f = f \text{ in } \Omega$$

and satisfying the supnorm estimate

$$\|\mathbf{T}f\|_{L^\infty(\Omega)} \leq C \|f\|_{L^\infty(\Omega)}$$

for a constant C independent of f.

Supnorm estimates for $\mathbf{T}f$: dimension 3

Let $\Omega = D_1 \times D_2 \times D_3$ be a product of open bounded domains with $C^{1,\alpha}$ boundary, $f = f_1 d\bar{z}_1 + f_2 d\bar{z}_2 + f_3 d\bar{z}_3$ a $\overline{\partial}$-closed form on Ω, with $f_j \in C^1(\Omega)$.

$$\begin{aligned}\mathbf{T}f := {} & T^1 f_1 + T^2 f_2 + T^3 f_3 - T^1 T^2 (\partial_{\bar{z}_1} f_2) \\ & - T^1 T^3 (\partial_{\bar{z}_1} f_3) - T^2 T^3 (\partial_{\bar{z}_2} f_3) + T^1 T^2 T^3 (\partial^2_{\bar{z}_1 \bar{z}_2} f_3)\end{aligned}$$

Supnorm estimates for $\mathbf{T}f$: dimension 3

Let $\Omega = D_1 \times D_2 \times D_3$ be a product of open bounded domains with $C^{1,\alpha}$ boundary, $f = f_1 d\bar{z}_1 + f_2 d\bar{z}_2 + f_3 d\bar{z}_3$ a $\bar{\partial}$-closed form on Ω, with $f_j \in C^1(\Omega)$.

$$\begin{aligned}\mathbf{T}f := {} & T^1 f_1 + T^2 f_2 + T^3 f_3 - T^1 T^2 (\partial_{\bar{z}_1} f_2) \\ & - T^1 T^3 (\partial_{\bar{z}_1} f_3) - T^2 T^3 (\partial_{\bar{z}_2} f_3) + T^1 T^2 T^3 (\partial^2_{\bar{z}_1 \bar{z}_2} f_3)\end{aligned}$$

Enough to estimate

$$T^1 T^2 T^3 (\partial^2_{\bar{z}_1 \bar{z}_2} f_3) = -\frac{1}{(2\pi i)^3} \int_{D_1 \times D_2 \times D_3} \frac{\partial^2_{\bar{\zeta}_1 \bar{\zeta}_2} f_3(\zeta_1, \zeta_2, \zeta_3)}{(\zeta_1 - z_1)(\zeta_2 - z_2)(\zeta_3 - z_3)}$$

Supnorm estimates for $\mathbf{T}f$: dimension 3

Define

$$G := |\zeta_2 - z_2|^2 |\zeta_3 - z_3|^2 + |\zeta_1 - z_1|^2 |\zeta_3 - z_3|^2 + |\zeta_1 - z_1|^2 |\zeta_2 - z_2|^2$$

We exploit

$$\frac{1}{(\zeta_1 - z_1)(\zeta_2 - z_2)(\zeta_3 - z_3)} = \frac{(\overline{\zeta_2 - z_2})(\overline{\zeta_3 - z_3})}{(\zeta_1 - z_1)\, G} + \frac{(\overline{\zeta_1 - z_1})(\overline{\zeta_3 - z_3})}{(\zeta_2 - z_2)\, G} + \frac{(\overline{\zeta_1 - z_1})(\overline{\zeta_2 - z_2})}{(\zeta_3 - z_3)\, G}$$

Supnorm estimates for $\mathbf{T}f$: dimension 3

By the previous identity and the fact that f is $\overline{\partial}$-closed

$$
\begin{aligned}
-(2\pi i)^3 T^1 T^2 T^3(\partial^2_{\bar{z}_1\bar{z}_2} f_3) &= \int_{D_1\times D_2\times D_3} \frac{\partial^2 f_1}{\partial\overline{\zeta}_2\partial\overline{\zeta}_3}\,\frac{(\overline{\zeta_2 - z_2})(\overline{\zeta_3 - z_3})}{(\zeta_1 - z_1)\,G} \\
&+ \int_{D_1\times D_2\times D_3} \frac{\partial^2 f_2}{\partial\overline{\zeta}_1\partial\overline{\zeta}_3}\,\frac{(\overline{\zeta_1 - z_1})(\overline{\zeta_3 - z_3})}{(\zeta_2 - z_2)\,G} \\
&+ \int_{D_1\times D_2\times D_3} \frac{\partial^2 f_3}{\partial\overline{\zeta}_1\partial\overline{\zeta}_2}\,\frac{(\overline{\zeta_1 - z_1})(\overline{\zeta_2 - z_2})}{(\zeta_3 - z_3)\,G} \\
&= I_1 + I_2 + I_3
\end{aligned}
$$

Supnorm estimates for $\mathbf{T}f$: dimension 3

By the previous identity and the fact that f is $\overline{\partial}$-closed

$$
\begin{aligned}
-(2\pi i)^3 T^1 T^2 T^3(\partial^2_{\overline{z}_1\overline{z}_2} f_3) &= \int_{D_1\times D_2\times D_3} \frac{\partial^2 f_1}{\partial\overline{\zeta}_2\partial\overline{\zeta}_3}\,\frac{(\overline{\zeta_2 - z_2})(\overline{\zeta_3 - z_3})}{(\zeta_1 - z_1)\,G} \\
&+ \int_{D_1\times D_2\times D_3} \frac{\partial^2 f_2}{\partial\overline{\zeta}_1\partial\overline{\zeta}_3}\,\frac{(\overline{\zeta_1 - z_1})(\overline{\zeta_3 - z_3})}{(\zeta_2 - z_2)\,G} \\
&+ \int_{D_1\times D_2\times D_3} \frac{\partial^2 f_3}{\partial\overline{\zeta}_1\partial\overline{\zeta}_2}\,\frac{(\overline{\zeta_1 - z_1})(\overline{\zeta_2 - z_2})}{(\zeta_3 - z_3)\,G} \\
&= I_1 + I_2 + I_3
\end{aligned}
$$

We now apply Stokes' theorem to move the derivatives away from the f_j

Supnorm estimates for $\mathbf{T}f$: dimension 3

After 2 applications of Stokes' theorem

$$
\begin{aligned}
I_3 = & \int_{\partial D_1 \times \partial D_2 \times D_3} f_3 \, \frac{(\overline{\zeta_1 - z_1})(\overline{\zeta_2 - z_2})}{(\zeta_3 - z_3)\, G} \\
& - \int_{\partial D_1 \times D_2 \times D_3} f_3 \, \frac{\partial}{\partial \bar{\zeta}_2} \left[\frac{(\overline{\zeta_1 - z_1})(\overline{\zeta_2 - z_2})}{(\zeta_3 - z_3)\, G} \right] \\
& - \int_{D_1 \times \partial D_2 \times D_3} f_3 \, \frac{\partial}{\partial \bar{\zeta}_1} \left[\frac{(\overline{\zeta_1 - z_1})(\overline{\zeta_2 - z_2})}{(\zeta_3 - z_3)\, G} \right] \\
& + \int_{D_1 \times D_2 \times D_3} f_3 \, \frac{\partial^2}{\partial \bar{\zeta}_1 \partial \bar{\zeta}_2} \left[\frac{(\overline{\zeta_1 - z_1})(\overline{\zeta_2 - z_2})}{(\zeta_3 - z_3)\, G} \right]
\end{aligned}
$$

The operator $\tilde{T}$

After Stokes' theorem

$$\mathbf{T}f = T^1 f_1 + T^2 f_2$$
$$- \frac{1}{(2\pi i)^2}\left(\int_{\partial D_1 \times D_2} \frac{(\overline{\zeta_1 - z_1}) f_2}{(\zeta_2 - z_2)|\zeta - z|^2} + \int_{D_1 \times \partial D_2} \frac{(\overline{\zeta_2 - z_2}) f_1}{(\zeta_1 - z_1)|\zeta - z|^2} \right.$$
$$\left. - \int_{D_1 \times D_2} \frac{(\overline{\zeta_1 - z_1}) f_1}{|\zeta - z|^4} - \int_{D_1 \times D_2} \frac{(\overline{\zeta_2 - z_2}) f_2}{|\zeta - z|^4} \right)$$

The operator $\tilde{T}$

After Stokes' theorem

$$\begin{aligned}\mathbf{T}f &= T^1 f_1 + T^2 f_2 \\ &- \frac{1}{(2\pi i)^2}\Bigg(\int_{\partial D_1 \times D_2} \frac{(\overline{\zeta_1 - z_1}) f_2}{(\zeta_2 - z_2)|\zeta - z|^2} + \int_{D_1 \times \partial D_2} \frac{(\overline{\zeta_2 - z_2}) f_1}{(\zeta_1 - z_1)|\zeta - z|^2} \\ &- \int_{D_1 \times D_2} \frac{(\overline{\zeta_1 - z_1}) f_1}{|\zeta - z|^4} - \int_{D_1 \times D_2} \frac{(\overline{\zeta_2 - z_2}) f_2}{|\zeta - z|^4}\Bigg)\end{aligned}$$

For the bidisc $D_1 \times D_2$, we can make sense of this operator even when $f_1, f_2 \in L^\infty(D_1 \times D_2)$. We thus define a new operator $\tilde{\mathbf{T}}$.