Effective algorithms for subelliptic multipliers

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$$V(f_1, \ldots, f_N) = \{0\} \iff 1 \in I_k \text{ for some } k$$

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Related to the subellipticity of the $\overline{\partial}$-Neumann problem on a domain Ω in $\mathbb{C}^{n+1}$ defined by

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For a general pseudoconvex domain Ω with $b\Omega$ smooth, the Kohn algorithm yields a sequence $I_1 \subseteq I_2 \subseteq \ldots$ in the ring C_p^∞.

Effectiveness of the holomorphic Kohn Algorithm

Consider the domain in $\mathbb{C}^3$ defined by

$$2\,\mathrm{Re}(\zeta) + |z|^2 + |w^3 + z^k w|^2 < 0.$$

The type at 0 is equal to 6, realized by $\gamma(t) = (0, t, 0)$.
The lower bound for ϵ depend on k. (Radical of order k).

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Triangular systems in $\mathbb{C}^n$ (Catlin-D'Angelo, 2010)

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Theorem 1 (F., 2019)

Let $\Omega \subset \mathbb{C}^{n+1}$ be a domain of finite type defined by

$$2\,\mathrm{Re}(z_{n+1}) + \sum_{j=1}^{N} |f_j(z)|^2 < 0,$$

where the f_j are polynomials of degree at most d. Then a subelliptic estimate holds at 0 with

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Machinery from commutative algebra:

- Effective Nullstellensatz (Kollár, Jelonek).
- Effective degree bounds for Gröbner bases (Dubé).
- Localization.

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For a domain $\Omega \subset \mathbb{C}^{n+1}$ with real analytic boundary $b\Omega$, we can write a defining function at $0 \in b\Omega$ as

$$2\,\mathrm{Re}(z_{n+1}) + \|f(z, z_{n+1})\|^2 - \|g(z, z_{n+1})\|^2$$

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Key point:

$$(1) \implies \lambda \geq c H_z \|f\|^2 .$$